mplrs: A scalable parallel vertex/facet enumeration code*

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Abstract

We describe a new parallel implementation, *mplrs*, of the vertex enumeration code *lrs* that uses the MPI parallel environment and can be run on a network of computers. The implementation makes use of a C wrapper that essentially uses the existing *lrs* code with only minor modifications. *mplrs* was derived from the earlier parallel implementation *plrs*, written by G. Roumanis in C++. *plrs* uses the Boost library and runs on a shared memory machine. In developing *mplrs* we discovered a method of balancing the parallel tree search, called budgeting, that greatly improves parallelization beyond the bottleneck encountered previously at around 32 cores. This method can be readily adapted for use in other reverse search enumeration codes. We also report some preliminary computational results comparing parallel and sequential codes for vertex/facet enumeration problems for convex polyhedra. The problems chosen span the range from simple to highly degenerate polytopes. For most problems tested, the results clearly show the advantage of using the parallel implementation *mplrs* of the reverse search based code *lrs*, even when as few as 8 cores are available. For some problems almost linear speedup was observed up to 1200 cores, the largest number of cores tested.
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1 Introduction

Given an $m \times n$ matrix $A = (a_{ij})$ and an m dimensional vector b, a *convex polyhedron*, or simply *polyhedron*, P is defined as:

$$P = \{x \in \mathbb{R}^n : b + Ax \geq 0\}. \tag{1}$$

This description of a polyhedron is known as an *H-representation*. A *polytope* is a bounded polyhedron. For simplicity in this article we will assume that the input data A, b define full dimensional polytopes. A point $x \in P$ is a *vertex* of P iff it is the unique solution to a subset of n inequalities solved as equations. The *vertex enumeration problem* is to output all vertices of a polytope P. This list of vertices gives us a *V-representation* of P. The reverse transformation, which takes a V-representation and computes its H-representation is known to be computationally equivalent via polarity. For further geometric background the reader is referred to Ziegler [19].

Vertex/facet enumeration problems find applications in many areas, of which we list a few here.[1] Early examples include computing the facets of correlation/cut polyhedra by physicists (see, e.g., [9, 12]) and current research in this area relates to detecting quantum behaviour in computers such as D-Wave. Research on facets of traveling salesman polytopes leads to important advances in branch-and-cut algorithms, see, e.g., [1]. For example, Chvátal local cuts are derived from facets of small

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[1]John White prepared a long list of applications which is available at [2]

TSPs and this idea is also seen in the small instance relaxations of Reinelt and Wenger [18]. Vertex enumeration is used to compute all Nash equilibria of bimatrix games and a code for this based on *lrs* is found at [2]. For minimizing extremely complicated concave functions vertex enumeration may be a last resort. This application shows the advantage of getting the output as a stream, most of which can be immediately discarded.

There are basically two algorithmic approaches to the vertex enumeration problem: the Fourier-Motzkin double description method (see, e.g., [19]) and pivoting methods such as Avis-Fukuda reverse search [4]. The double description method involves inserting the half spaces from the H-representation sequentially and updating the list of vertices that they span. This sequential method does not seem easy to parallelize. Readily available codes for this method include *cddr+* [13], *ppl_lcdd* [8], PORTA [10] and Qhull [17]. The reverse search method for vertex enumeration was implemented as *lrs* [2].

From the outset it was realized that reverse search was eminently suitable for parallelization. The first such code, *prs* was developed by Ambros Marzetta using his ZRAM parallelization platform, as described in [7] and available online at [16]. In this case the parallelization was built into the *lrs* code itself leading to problems of maintenance and upgrading as newer parallel libraries developed.

The *lrs* code is rather complex and has been under development for over twenty years incorporating a multitude of different functions. It has been used extensively and its basic functionality is very stable. Directly adding parallelization code to such legacy software is extremely delicate and can easily produce bugs that are difficult to find. Therefore later approaches avoided this problem by implementing the parallelization as a separate layer with very few changes to *lrs* itself. This allowed independent development of both parallelization ideas and basic improvements in the underlying code, at the cost of certain overheads that we discuss later. It relies on the fact that (a) in reverse search subtrees can be enumerated completely independently of each other, and (b) a list of subtrees can be obtained by running *lrs* down to some fixed depth in a first phase.

The concept was first tested by a shell script, *tlrs*, developed by John White in 2009. Here the parallelization was achieved by scheduling independent processes for subtrees via the shell. Although good speedups were obtained several limitations of this approach materialized as the number of processors available increased. In particular job control becomes a major issue: there is no single controlling process.

To overcome these limitations the first author and Gary Roumanis developed *plrs* [6]. This code is a C++ wrapper that compiles in the original *lrslib* library essentially maintaining the integrity of the underlying *lrs* code. The parallelization was achieved by multithreading using the Boost library and was designed to run on shared memory machines with little user interaction. In this way users almost automatically get improved performance as they upgrade their workstations to include more processors. Experience gained with the *plrs* code, especially in the area of load balancing, was the starting point for *mplrs*.

The remainder of the paper is organised as follows. We begin with background on reverse search, vertex enumeration and *lrs*. A brief description of *plrs* and its strengths and weaknesses will then be discussed in Section 2.2. This is followed in Section 3 by a discussion of various parallelization strategies that could be employed to manage the load balancing problem. In Section 4 we discuss the implementation of *mplrs* and describe its features. Finally in Section 5 we give some test results on a wide range of inputs, comparing 5 codes: *cddr+*, *ppl_lcdd*, *lrs*, *plrs* and *mplrs*.

2 Background

2.1 Reverse search and *lrs*

Reverse search is a technique for generating large, relatively unstructured, sets of discrete objects. We give an outline of the method here and refer the reader to [4, 5] for further details. In its most basic form, reverse search can be viewed as the traversal of a spanning tree, called the reverse search tree T, of a graph $G = (V, E)$ whose nodes are the objects to be generated. Edges in the graph are specified by an adjacency oracle, and the subset of edges of the reverse search tree are determined by an auxiliary function, which can be thought of as a local search function f for an optimization problem defined on the set of objects to be generated. One vertex, v^*, is designated as the *target* vertex. For

every other vertex $v \in V$ repeated application of f must generate a path in G from v to v^*. The set of these paths defines the reverse search tree T, which has root v^*.

A reverse search is initiated at v^*, and only edges of the reverse search tree are traversed. When a node is visited the corresponding object is output. Since there is no possibility of visiting a node by different paths, the nodes are not stored. Backtracking can be performed in the standard way using a stack, but this is not required as the local search function can be used for this purpose. This implies two critical features that are essential for effective parallelization. Firstly it is not necessary to store more than one node of the tree at any given time and no database is required for visited nodes. Secondly it is possible to *restart* the enumeration process from any given node in the tree using only a description of this one node.

In the basic setting described here a few properties are required. Firstly, the underlying graph G must be connected and an upper bound on the maximum vertex degree, Δ, must be known. The performance of the method depends on G having Δ as low as possible. The adjacency oracle must be capable of generating the adjacent vertices of some given vertex v sequentially and without repetition. This is done by specifying a function $Adj(v, j)$, where v is a vertex of G and $j = 1, 2, ..., \Delta$. Each value of $Adj(v, j)$ is either a vertex adjacent to v or null. Each vertex adjacent to v appears precisely once as j ranges over its possible values. For each vertex $v \neq v^*$ the local search function $f(v)$ returns the tuple (u, j) where $v = Adj(u, j)$ such that u is v's parent in T. Pseudocode is given in Algorithm 1. Note that the vertices are output as a continuous stream. For convenience later, we do not output the root vertex v^* in the pseudocode shown.

Algorithm 1 Generic Reverse Search

```
 1: procedure RS(v*, Δ, Adj, f)
 2:     v ← v*   j ← 0   depth ← 0
 3:     repeat
 4:         while j < Δ do
 5:             j ← j + 1
 6:             if f(Adj(v, j)) = v then                    ▷ forward step
 7:                 v ← Adj(v, j)
 8:                 j ← 0
 9:                 depth ← depth + 1
10:                 output v
11:             end if
12:         end while
13:         if depth > 0 then                               ▷ backtrack step
14:             (v, j) ← f(v)
15:             depth ← depth − 1
16:         end if
17:     until depth = 0 and j = Δ
18: end procedure
```

To apply reverse search to vertex enumeration we first make use of *dictionaries*, as is done for the simplex method of linear programming. To get a dictionary for (1) we add one new nonnegative variable for each inequality:

$$x_{n+i} = b_i + \sum_{j=1}^{n} a_{ij} x_j, \quad x_{n+i} \geq 0 \quad i = 1, 2, ..., m.$$

These new variables are called *slack variables* and the original variables are called *decision variables*.

In order to have any vertex at all we must have $m \geq n$, and normally m is significantly larger than n, allowing us to solve the equations for various sets of variables on the left hand side. The variables on the left hand side of a dictionary are called *basic*, and those on the right hand side are called *non-basic* or, equivalently, *co-basic*. We use the notation $B = \{i : x_i \text{ is basic}\}$ and $N = \{j : x_j \text{ is co-basic}\}$.

A *pivot* interchanges one index from B and N and solves the equations for the new basic variables. A *basic solution* from a dictionary is obtained by setting $x_j = 0$ for all $j \in N$. It is a *basic feasible solution (BFS)* if $x_j \geq 0$ for every slack variable x_j. A dictionary is called *degenerate* if it has a slack basic variable $x_j = 0$. As is well known, each BFS defines a vertex of P and each vertex of P can be represented as one or more (in the case of degeneracy) BFSs.

Next we define the relevant graph $G = (V, E)$ to be used in Algorithm 1. Each node in V corresponds to a BFS and is labelled with the cobasic set N. Each edge in E corresponds to a pivot between two BFSs. Formally we may define the adjacency oracle as follows. Let B and N be index sets for the current dictionary. For $i \in B$ and $j \in N$

$$Adj(N, i, j) = \begin{cases} N \setminus \{j\} \cup \{i\} & \text{if this gives a feasible dictionary} \\ \emptyset & \text{otherwise.} \end{cases}$$

The target v^* for the reverse search is found by solving a linear program over this dictionary with any objective function $z = c^T x$ that defines a unique optimum vertex. We use the objective function z and a non-cycling pivot selection rule to define the local search function f. In the case of *lrs*, Bland's least subscript rule for selecting the variable which enters the basis and a lexicographic ratio test to select the leaving variable are used. The lexicographic rule simulates a simple polytope which greatly reduces degeneracy. *lrs* is an implementation of Algorithm 1 in exact rational arithmetic using Adj, f, and v^* as just described.

2.2 Parallelization and *plrs*

The *plrs* code was described in detail in [6] along with the rather minor changes required to the legacy code *lrs*. Here we give a generic description of the parallelization which is by nature somewhat oversimplified. We will use as an example the tree shown in Figure 1 which shows the first two layers of the reverse search for the problem *mit* that will be described in Section 5. The weight on each node is the number of nodes in the subtree that it roots. The root of the tree is in the centre and its weight shows that the tree contains 1375608 nodes, the number of cobases generated by *lrs*. At depth 2 there are 35 nodes but, of these, just the four underlined nodes contain collectively about 58% of the total number of tree nodes.

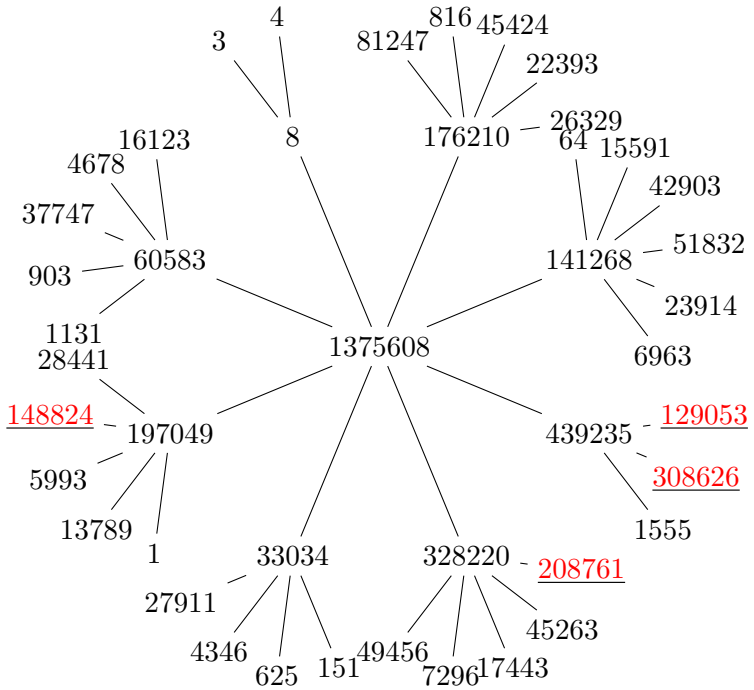


Figure 1: Number of nodes in subtrees at depth 2 for `mit.ine`

The method implemented in *plrs* proceeds in three phases. In the first phase we generate the reverse search tree T down to a fixed depth, *init_depth*, reporting all nodes to the output stream. In

addition, the nodes of the tree with depth equal to *init_depth* which are not leaves of T are stored in a list L.

In the second phase we schedule subtree enumeration for nodes in L using a user supplied parameter *max_threads* to limit the number of parallel processes. For subtree enumeration we use *lrs* with a slight modification to its earlier described restart feature. Normally, in a restart, *lrs* starts at a given restart node at its given depth and computes all remaining nodes in the tree T. The simple modification is to supply a depth of zero with the restart node so that the search terminates when trying to backtrack from this node.

When the list L becomes empty we move to Phase 3 in which the threads terminate one by one until there are no more running and the procedure terminates. In both Phase 2 and Phase 3 we make use of a *collection process* which concatenates the output from the threads into a single output stream. It is clear that the only interaction between the parallel threads is the common output collection process. The only signalling required is when a thread initiates or terminates a subtree enumeration.

Let us return to the example in Figure 1. Suppose we set the $init\_depth = 2$ and $max\_threads = 12$. A total of 35 nodes are found at this depth. 34 are stored in L and the other, being a leaf, is ignored. The first 12 nodes are removed from L and scheduled on the 12 threads. Each time a subtree is completely enumerated the associated thread receives another node from L and starts again. When L is empty the thread is idle until the entire job terminates. To visualize the process refer to Figure 2. In this case we have set $init\_depth = 3$ to obtain a larger L. The vertical axis shows thread usage and the horizontal axis shows time. Phase 1 is so short - less than one second - that it does not appear. Phase 2 lasts about 50 seconds, when all 12 threads are busy. Phase 3 lasts the remaining 70 seconds as more and more threads become idle. If we add more cores, only Phase 2 will profit. Even with very many cores the running time will not drop below 70 seconds and so this technique does not scale well. In comparing Figures 1 and 2 we see that the few large subtrees create an undesirably long Phase 3. Going to a deeper initial depth helps to some extent, but this eventually creates an extremely long list L with subsequent increase in overhead (see [6] for more details). Nevertheless *plrs* performs very well with up to about 32 parallel threads, as we will see in Section 5.

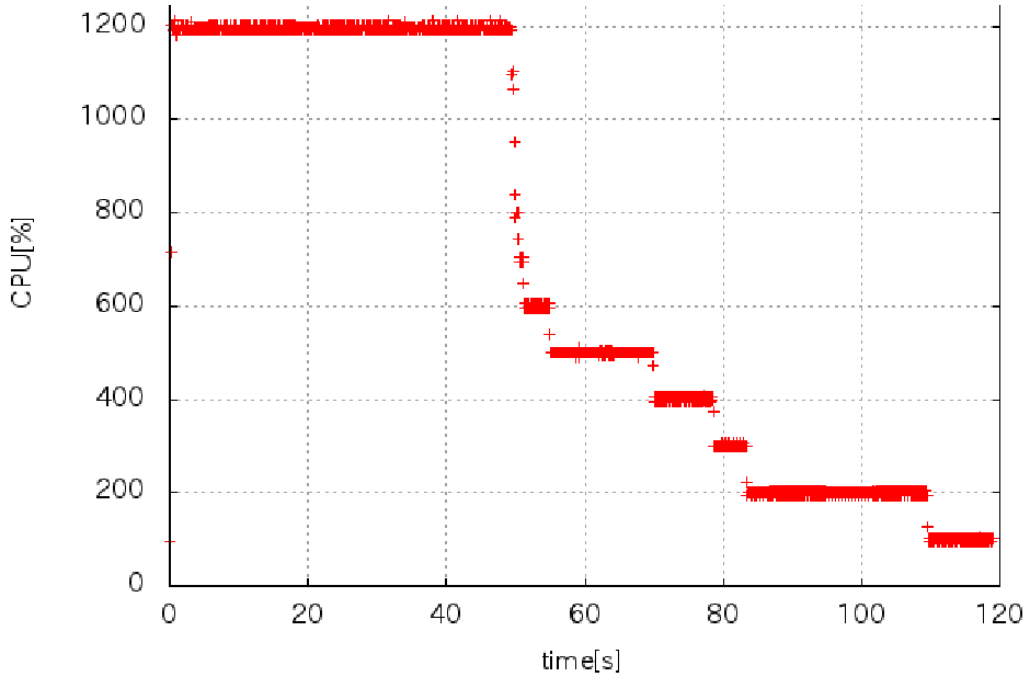


Figure 2: Processor usage by *plrs* on problem *mit* on a 12 core machine, $init\_depth = 3$

In analyzing this method we observe that in Phase 1 there is no parallelization, in Phase 2 all available cores are used, and in Phase 3 the level of parallelization drops monotonically as threads terminate. Looking at the overhead compared with *lrs* we see that this almost entirely consists of the amount of time required to restart the reverse search process. In this case it requires the time to

pivot the input matrix to a given cobasis, which is not neglible. However a potentially greater cost occurs when L is empty and threads are idle. As the number of available processors increase this cost goes up, but the overhead of restarting remains the same, for given fixed *init_depth*. This leads to conflicting issues in setting the critical *init_depth* parameter. A larger value implies that:

- only a single thread is working for a longer time
- the list L will typically be larger requiring more overhead in restarts, but
- the time spent in Phase 3 will typically be reduced.

The success in parallelization clearly depends on the structure of the tree T. In the worst case it is a path and no parallelization occurs in Phase 2. Therefore in the worst case we have no improvement over *lrs*. In the best case the tree is balanced so that the list L can be short reducing overhead and all threads terminate at more or less the same time. Success therefore heavily depends on the structure of the underlying enumeration problem.

3 Parallelization Strategies

plrs generates subproblems in an initial phase based on a user supplied *init_depth* parameter. This tends to perform best on balanced trees which, in practice, seem rather rare. In *plrs*, workers (except the initial Phase 1 worker) always finish the subproblem that they are assigned. However, there is no guarantee that subproblems have similar sizes and as we have seen they can differ dramatically. As we saw earlier, this can lead to a major loss of parallelism after the queue L becomes empty. We explored various more sophisticated strategies in developing *mplrs* and these are described in this section. In particular, we focus on

- estimating the size of subproblems to improve scheduling and create reasonably-sized problems,
- dynamic creation of subproblems, where we can split subproblems at any time instead of only during the initial phase,
- using budgets for workers, who return after exploring a budgeted number of nodes adding unfinished subproblems to L.

3.1 Subtree Estimation

A glance at Figure 1 shows the problem with using a fixed initial depth to generate the subtrees for L: the tree mass is concentrated on very few nodes. Obviously increasing *init_depth* would limit the size of the large subtrees, but this would greatly increase the number of jobs in L, increasing the restart overhead. Since *lrs* has the capability to estimate subtree size we tried two approaches based on it: priority scheduling and iterative deepening.

Estimation is possible for vertex enumeration by reverse search using Hall-Knuth estimation [14]. From any node a child can be chosen at random and by continuing in the same way a random path to a leaf is constructed. This leads to an unbiased estimate of the subtree size from the initial node. Various methods lead to lower variance, see [3].

The first use of estimation we tried was in priority scheduling. As is well known, a good heuristic for list scheduling on multiprocessors is to schedule the longest jobs first. Referring again to Figure 1 we see that we should schedule those four heaviest subtrees at the start of Phase 2. Since we do not have the exact values of the subtree sizes we decided to use the estimation function as a proxy. We then scheduled jobs from L in a list decreasing manner by estimated tree size.

A second idea we tried was iterative deepening. We start by setting a threshold value, say k, for maximum estimated subtree size. Once a node at *init_depth* is encountered an estimate of its subtree size is made. If this exceeds k then we continue to the next layer of the tree and estimate the subtree sizes again, repeatedly going deeper in the tree for subtrees whose estimates exceed k. In this way all nodes returned to L will have estimated subtree sizes smaller than k.

The results from these two approaches were mixed. There are two negative points. One is that Hall-Knuth estimates have very high variance, and the true value tends to be larger than the estimate in probability. So very large subtrees receiving small estimates would not be scheduled first in priority scheduling and would not be broken up by iterative deepening. Secondly the nodes visited during the random probes represent overhead, as these nodes will all be visited again later. In order to improve the quality of the estimate a large number of probes need to be made, increasing this overhead.

Nevertheless this seems to be an interesting area of research. Newer more reliable estimation techniques that do not result in much overhead, such as the on-the-fly methods of [11] and [15], may greatly improve the effectiveness of these approaches.

3.2 Dynamic Creation of Subproblems

As we saw in Section 2.2, *plrs* creates new subproblems only during the initial phase. We can think in terms of one boss, who creates subproblems in Phase 1, and a set of workers who start work in Phase 2 and each work on a single subproblem until it is completed. However, there is no reason why an individual worker cannot send some parts of its search tree back to L without exploring them.

A simple example of this is to implement a *skip* parameter. This is set at some integer value $t > 1$ and subtrees rooted at every t-th node explored are sent back to L without exploration. The boss can set the *skip* parameter dynamically when allocating work from L. If L is getting dangerously small, then a small value is set. Conversely if L is very large an extremely large value is set.

We implemented this idea but did not get good results. When the *skip* parameter is set then all subtrees are split into smaller pieces, even the small subtrees, which is undesirable. When *skip* is too small, the list L quickly becomes unmanageably large with very high overhead. It seemed hard for the boss to control the size of L by varying the size of the parameter, due to latency effects.

3.3 Budgeted Subproblems

The final and most successful approach involved limiting the amount of work a worker could do before being required to quit. Each worker is given a *budget* which is the maximum number of nodes that can be visited. Once this budget is exceeded the worker backtracks to the root of its subtree returning all unfinished subproblems. These consist of all unexplored children of nodes in the backtrack path. This has several advantages. Firstly, if the subtree has size less than the budget (typically 5000 nodes in practice) then the entire subtree is evaluated without additional creation of overhead. Secondly, each large subtree automatically gets split up. By including all unexplored subtrees back to the root a variable number of jobs will be added to L. A giant subtree will be split up many times. For example, the subtree with 308626 nodes in Figure 1 will be split over 600 times, providing work for idle workers. We can also change the budget dynamically to obtain different effects. If the budget is set to be small we immediately create many new jobs for L. If L grows large we can increase the budget: since most subtrees will be below the threshold the budget is not used up and new jobs are not created.

Budgeting can be introduced to the generic reverse search procedure of Algorithm 1 as follows. When calling the reverse search procedure we now supply three additional parameters:

- *start_vertex* is the vertex from which the reverse search should be initiated and replaces v^*.
- *max_depth* is the depth at which forward steps are terminated.
- *max_cobases* is the number of nodes to generate before terminating and reporting unexplored subtrees.

Both *max_depth* and *max_cobases* are assumed to be positive, for otherwise there is no work to do. The modified algorithm is shown in Algorithm 2.

Comparing Algorithm 1 and Algorithm 2 we note several changes. Firstly an integer variable *count* is introduced to keep track of how many tree nodes have been generated. Secondly a flag *unexplored* is introduced to distinguish the tree nodes which have not been explored and which are to be placed on L. It is initialized as $false$ on line 4. The flag is set to *true* in line 13 if either the budget of *max_cobases* has been exhausted or a depth of *max_depth* has been reached. In any case, each node

Algorithm 2 Budgeted Reverse Search

```
 1: procedure BRS(start_vertex, Δ, Adj, f, max_depth, max_cobases )
 2:     j ← 0   v ← start_vertex   count ← 0   depth ← 0
 3:     repeat
 4:         unexplored ← false
 5:         while j < Δ and unexplored = false  do
 6:             j ← j + 1
 7:             if f(Adj(v, j)) = v then                          ▷ forward step
 8:                 v ← Adj(v, j)
 9:                 j ← 0
10:                 count ← count + 1
11:                 depth ← depth + 1
12:                 if count ≥ max_cobases or depth = max_depth then   ▷ budget is exhausted
13:                     unexplored ← true
14:                 end if
15:                 put_output (v, unexplored)
16:             end if
17:         end while
18:         if depth > 0 then                                     ▷ backtrack step
19:             (v, j) ← f(v)
20:             depth ← depth − 1
21:         end if
22:     until depth = 0 and j = Δ
23: end procedure
```

encountered on a forward step is output via the routine *put_output* on line 15. In single processor mode the output is simply sent to the output file with a flag added to unexplored nodes. In multi-processor mode, the output is synchronized and unexplored nodes are returned to L (cf. Section 4).

Backtracking is as in Algorithm 1. After each backtrack step the *unexplored* flag is set to $false$ in line 4. If the budget constraint has been exhausted then *unexplored* will again be set to *true* in line 13 after the first forward step. In this way all unexplored siblings of nodes on the backtrack path to the root are flagged and placed on L. If the budget is not exhausted then forward steps continue until either it is, *max_depth* is reached, or we reach a leaf.

To output all nodes in the subtree of T rooted at v we set $start\_vertex = v$, $max\_cobases = +\infty$ and $max\_depth = +\infty$. So if $v = v^*$ this reduces to Algorithm 1. For budgeted subtree enumeration we set *max_cobases* to be the worker's budget. To initialize the parallelization process we will generate the tree T down to a a small fixed depth with a small budget constraint in order to generate a lot of subtrees. We then increase the budget constraint and remove the depth constraint so that most workers will finish the tree they are assigned without returning any new subproblems for L. The details are given in Section 4.1.

4 Implementation of *mplrs*

The primary goals of *mplrs* were to move beyond single, shared-memory systems to clusters and improve load balancing when a large number of cores is available. The implementation uses MPI, and starts a user-specified number of processes on the cluster. One of these processes becomes the *master*, another becomes the *consumer*, and the remaining processes are *workers*.

The master process is responsible for distributing the input file and parametrized subproblems to the workers, informing the other processes to exit at the appropriate time, and handling checkpointing. The consumer receives output from the workers and produces the output file. The workers receive parametrized subproblems from the master, run the *lrs* code, send output to the consumer, and return unfinished subproblems to the master if the budget has expired.

4.1 Master process

The master process begins by sending the input to all workers, which may not have a shared filesystem. In *mplrs*, *Adj* and f are defined as in Section 2.1 and so it suffices to send the input polyhedron. Pseudocode for the master is given in Algorithm 3.

Since we begin from a single *start_vertex*, the master chooses an initial worker and sends it the initial subproblem. We cannot yet proceed in parallel, so the master uses user-specified (or very small default) initial parameters *init_depth* and *max_cobases* to ensure that this worker will return (hopefully many) unfinished subproblems quickly. The master then executes its main loop, which it continues until no workers are running and the master has no unfinished subproblems. Once the main loop ends, the master informs all processes to finish. The main loop performs the following tasks.

- if there is a free worker and the master has a subproblem, subproblems are sent to workers.
- we check if any workers are finished, mark them as free and receive their unfinished subproblems.

Algorithm 3 Master process

```
1: procedure MPRS(start_vertex, Δ, Adj, f, init_depth, max_depth, max_cobases, lmin, lmax,
   scale, num_workers)
2:     Send (Δ, Adj, f) to each worker
3:     Create empty lists L, working
4:     size ← num_workers + 2
5:     Send (start_vertex, init_depth, max_cobases) to worker 1
6:     Add 1 to list working
7:     Mark 1 as working
8:     while L is not empty or working is not empty do
9:         while L is not empty and some worker not marked as working do
10:            if |L| < size · lmin then
11:                maxd ← max_depth
12:            else
13:                maxd ← ∞
14:            end if
15:            if |L| > size · lmax then
16:                maxc ← scale · max_cobases
17:            else
18:                maxc ← max_cobases
19:            end if
20:            Remove next element start from L
21:            Send (start, maxd, maxc) to first free worker i
22:            Add i to working
23:            Mark i as working
24:        end while
25:        for i in working do
26:            Check for new message unfinished from i
27:            if incoming message unfinished from i then
28:                Join list unfinished to L
29:                Remove i from working
30:                Unmark i as working
31:            end if
32:        end for
33:    end while
34:    Send terminate to all processes
35: end procedure
```

Using reasonable parameters is critical to achieving good parallelization. As described in Section 3.3 this is done dynamically by observing the size of L. We use the parameters $lmin$, $lmax$ and $scale$. Initially, to create a reasonable size list L, we set $max\_depth = 2$ and $max\_cobases = 50$. Therefore the initial worker will generate subtrees at depth 2 until 50 nodes have been visited and then backtrack. Additional workers are given the same agressive parameters until L grows larger than $lmax$ times the number of processors. We now multiply the budget by $scale$ and remove the max_depth constraint. Currently $scale = 100$ so workers will not generate any new subproblems unless their tree has at least 5000 nodes. If the length of L drops below this bound we return to the earlier value of $max\_cobases = 50$ and if it drops below $lmin$ times the size of L we reinstate the max_depth constraint. The current default is to set $lmin = lmax = 3$. In Section 4.4 we show an example of how the length of L typically behaves with these parameter settings.

4.2 Workers

The worker processes are simpler – they receive the problem at startup, and then repeat their main loop: receive a parametrized subproblem from the master, work on it subject to the parameters, send the output to the consumer, and send unfinished subproblems to the master if the budget is exhausted.

Algorithm 4 Worker process

```
1:  procedure WORKER
2:      Receive (Δ, Adj, f) from master
3:      while true do
4:          Wait for message from master
5:          if message is terminate then
6:              Exit
7:          end if
8:          Receive (start_vertex, max_depth, max_cobases)
9:          Call BRS(start_vertex, Δ, Adj, f, max_depth, max_cobases)
10:         Send list of unfinished vertices to master
11:         Send output list to consumer
12:     end while
13: end procedure
```

4.3 Consumer process

The consumer process in *mplrs* is the simplest. The workers send output to the consumer in exactly the format it should be output (i.e. this formatting is done in parallel). The consumer simply sends it to an output file, or prints it if desired. By synchronizing output to a single destination, the consumer delivers a continuous output stream to the user in the same way as *lrs* does.

Algorithm 5 Consumer process

```
1: procedure CONSUMER
2:     while true do
3:         Wait for incoming message
4:         if message is terminate then
5:             Exit
6:         end if
7:         Output this message
8:     end while
9: end procedure
```

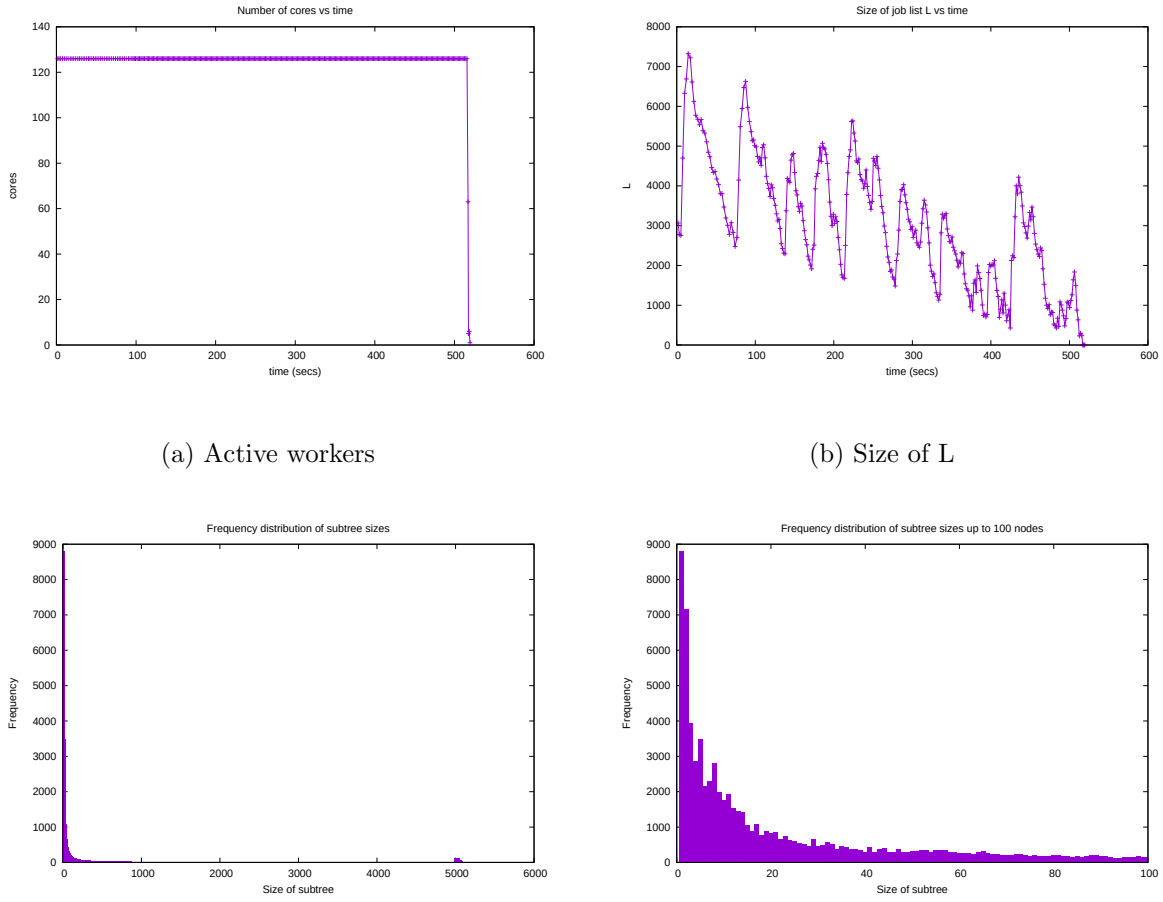
4.4 Histograms

There are additional features supported by *mplrs* that are minor additions to Algorithms 3–5. We introduce *histograms* in this subsection, before proceeding to checkpoints in Section 4.5.

When desired, *mplrs* can provide a variety of information in a histogram file. Periodically, the master process adds a line to this file, containing the following information.

- real time in seconds since execution began.
- the number of workers marked as working.
- the current size of L (number of subproblems the master has).

We use this histogram file with `gnuplot` to produce plots that help understand how much parallelization is achieved over time, which helps tune parameters. Examples of the resulting output are shown in Figure 3.



(a) Active workers

(b) Size of L

(c) Distribution of subproblem sizes

(d) Distribution of *small* subproblem sizes

Figure 3: Histograms for mit71 with 128 processes

In Figure 3a, we see that workers are generally not idle. This is a requirement for effective parallelization, but it is not sufficient: if the job queue is very large the overhead required to start jobs will dominate and performance is lost. To get information on this the second histogram, Figure 3b, is of use. This plot gives the size of L, the number of subproblems held by the master.

In *mplrs*, L is implemented as a stack. When $|L|$ falls to a value for the first time, a new (relatively high in the tree) subproblem is examined for the first time. If this new subproblem happens to be large, the size of L can grow dramatically due to the budget being exhausted by the assigned worker. The choice of parameters greatly affects the rate at which new subproblems are created.

These histograms can also be used to visualize the overall progress of the run in real time to see if the parameters are reasonable. The pattern shown in Figure 3b was typical of the cases we examined and provides a rough estimate of the remaining run time if viewed in real time.

A third type of histogram, subtree size, can also be produced as shown in Figure 3c. This gives the frequency of the sizes of all subtrees whose roots were stored in the list L, which in this case contained a total of 116,491 subtree roots. We see that for this problem the vast majority of subtrees are extremely small. The detail of this is shown in Figure 3d. These small subtrees could have been enumerated more quickly than their restart cost alone – if they could have been identified quickly. This is an interesting research problem. After about 60 nodes the distribution is quite flat until the small hump occuring at 5000 nodes. This is due to the budget limit of 5000 causing a worker to terminate. The hump continues slightly past 5000 nodes reflecting the additional nodes the worker visits on the backtrack path back to the root. It is interesting that most workers completely finish their subtrees and only very few actually hit the budget constraint. Histograms such as these may be of interest for theoretical analysis of the budgeting method. For example, the shape of the histogram may suggest an appropriate random tree model to study for this type of problem.

4.5 Checkpointing

An important feature of *mplrs* is the ability to checkpoint and restart execution with potentially different parameters or number of processes. This allows, for example, users to tune parameters over time using the histogram file, without discarding initial results. It is also very useful for very large jobs if machines need to be turned off for any reason or if new machines become available.

Checkpointing is easy to implement in *mplrs* but to be effective it depends heavily on the *max_cobases* option being set. Workers are never aware of checkpointing or restarting – as in Algorithm 4 they simply use *lrs* to solve given subproblems until their budget runs out. When the master wishes to checkpoint, it ceases distribution of new subproblems and tells workers to terminate. Once all workers have finished and returned any unfinished subproblems, the master informs the consumer of a checkpoint. The consumer then sends various counting statistics to the master, which saves these statistics and L in a *checkpoint file*. Note that if *max_cobases* is not set then each worker must completely finish the subtree assigned, which may take a very long time!

When restarting from a checkpoint file, the master reloads L from the file instead of distributing the initial subproblem. It informs the consumer of the counting statistics and then proceeds normally. Previous output is not re-examined: *mplrs* assumes that the checkpoint file is correct.

5 Performance

We describe here some experimental results for the three codes described in this paper and two codes based on the double description method: *cddr+* [13] and *ppl_lcdd* [8].

The input files are described in Table 1 and range from simple polyhedra to extremely degenerate polyhedra. They are contained in the lrslib-061 distribution available at [2] in subdirectory `lrslib-061/ine/test-061`. The problems are:

- *mit*, *mit*71: configuration polytopes created by G. Garbulsky [9]
- fq48-19: related to the traveling salesman problem for $n = 5$, created by F. Quondam
- $m6.ine$: the metric polytope for $n = 6$ [12]
- *cp*6: the cut polytope for $n = 6$ solved in the 'reverse' direction: from an H-representation to a V-representation [12]
- *c*40-20: a cyclic polytope with 40 vertices in R^{20} [19]
- *perm*10: permutahedron for $n = 10$ written in its standard formulation [19]
- *bv*7: extended formulation for the permutahedron for $n = 7$ based on the Birkhov polytope [19]

Except for *cp*6, the time limit set was one week (604,800 seconds). Programs *cddr+*, *lrs*, *ppl_lcdd* were used with no parameters. For *mplrs* and *plrs* we used the default settings (see User's guide [2] for details):

- *plrs*: -id 4
- *mplrs*: -id 2, -lmin 3 -maxc 50 -scale 100

The tests were performed using two computers:

- *mai20*: 2x Xeon E5-2690 (10-core 3.0GHz), 128GB memory, 3TB hard drive (20 cores)
- *mai128*: 4 nodes, each containing: 2x Opteron 6376 (16-core 2.3GHz), 32GB memory, 500GB hard drive (128 cores in total)

Name	Input			Output	
	H/V	m	n	V/H	size
bv7	H	69	57	5040	867K
c40-20	V	40	21	40060020	15.6G
fq48-19	H	48	19	119184	8.7M
m6	H	80	16	554	41K
mit71	H	71	61	3149579	1.1G
mit	H	729	9	4862	196K
perm10	H	1023	11	3628800	127M
cp6	H	368	16	32	1.6K

Name	*lrs*			cddr+	ppl
	bases	depth	secs	secs	secs
bv7	84707280	17	8848	*	578
c40-20	20030010	19	9870	*	*
fq48-19	7843390	24	367	124	1355
m6	6651872	32	496	1	0.1
mit71	57613364	20	21356	*	260347
mit	1375608	101	558	404	40644
perm10	3628800	45	2457	*	*
cp6	4844923002	153	1762156	1463829	>6570000

Table 1: Polyhedra tested and baseline times: *=killed after 604800 secs (*mai20*)

Table 2 contains results for low scale parallelization and were run on the single workstation *mai20*. With up to 12 cores available, *plrs* usually outperforms *mplrs*. Table 3 contains results for medium scale parallelization and were run on *mai128*. Since this is a cluster of four 32-core nodes, we cannot run *plrs* on either 64 or 128 cores. The column for $mt = 128$ gives a speedup over *lrs* as well as a speedup over $mt = 16$. The speedups on *lrs* clearly depend on the problem type. *c*40-20 with its huge numbers and output size gives the poorest speedup over *lrs* for both *mplrs* and *plrs*, perhaps due to the large overhead in pivoting a huge dictionary during restart. This is evidenced by the fact that *mplrs* with 4 cores (ie. 2 workers) actually takes twice as long as *lrs* on this problem! However, note that the list L, and hence the total restart overhead, is mostly *independent* of the number of cores available. So when comparing *mplrs* with 128 cores to 16 cores we find a respectable speedup of 6.7 times. The overhead of the restart is effectively amortized over the number of workers leading to the scalability of our method.

Table 4 contains results on the large scale parallelization obtained by Kazuki Yoshizoe using the *Tsubame2* supercomputer at the Tokyo Institute of Technology. The hardest problem solved was *cp*6, the 6 point cut polytope solved in the reverse direction, which is extremely degenerate. Its more than 4.8 billion bases span just 32 vertices! Normally such polytopes would be out of reach for pivoting algorithms. In fact on *mai20* it it took *lrs* just under 3 weeks to complete the computation, about four

Name	*lrs* secs/size	*mt*=4 secs/speedup		*mt*=8 secs/speedup		*mt*=12 secs/speedup	
		mplrs	*plrs*	*mplrs*	*plrs*	*mplrs*	*plrs*
bv7	8848	5232	2444	1762	1249	1042	827
	867K	1.7	3.6	5.0	7.1	8.5	10.7
c40-20	9870	19405	3779	6314	2616	3830	2276
	15.6G	0.5	3.6	1.6	4.7	2.6	4.9
fq48-19	367	147	102	49	53	41	37
	8.7M	2.5	3.6	7.5	6.9	9.0	9.9
m6	496	277	151	92	86	56	63
	41K	1.8	3.3	5.4	5.8	8.9	7.8
mit71	21356	12096	6492	3996	3311	2415	2243
	1.1G	1.8	3.3	5.3	6.5	8.8	9.5
mit	558	302	161	101	94	62	71
	196K	1.8	3.5	5.5	5.9	9.0	7.9
perm10	2457	1528	741	499	474	314	381
	127M	1.6	3.3	4.9	5.2	7.8	6.4
cp6	1762156	968550	1007184	331235	565915	199501	434390
	1.6K	1.8	1.7	5.3	3.1	8.8	4.1

Table 2: Small scale parallelization: *mt*=no. of processors (*mai20*)

Name	*lrs* secs/size	*mt* = 16 secs/su		*mt*=32 secs/su		*mt*=64 secs/su	*mt*=128 secs/sus	
		mplrs	*plrs*	*mplrs*	*plrs*	*mplrs*	*mplrs*	
bv7	14821	1458	1197	946	851	458	228	
	867K	10.1	12.4	15.7	17.4	32.3	65	6.4
c40-20	15671	7829	3955	4870	3801	3346	1160	
	15.6G	2.0	4.0	3.2	4.1	4.6	13.5	6.7
fq48-19	450	41	40	25	29	12	7	
	8.7M	11.0	11.3	18	15.5	37.5	64.2	5.9
m6	840	80	94	50	86	25	14	
	41K	10.5	8.9	16.8	9.8	33.6	60	5.7
mit71	39078	3438	3396	2160	2351	1049	520	
	1.1G	11.4	11.5	18.1	16.6	37.2	75	6.6
mit	1009	94	115	62	111	38	22	
	196K	10.7	8.8	16.3	9.1	26.6	45.9	4.3
perm10	4163	453	590	282	548	179	88	
	127M	9.2	7.1	14.8	7.6	23.3	47.3	5.1
cp6	3081440	291361	393288	177132	264964	86569	42323	
	1.6K	10.6	7.8	17.4	11.6	35.6	72.8	6.9

Table 3: Medium scale parallelization: *mt*=no. of processors (*mai128*)
(*mt* = 128 speedups are on both *lrs* and *mt* = 16)

Name	*mplrs*						
	mt=12	*mt*=36	*mt*=72	*mt*=144	*mt*=300	*mt*=600	*mt*=1200
cp6				20383	9782	4913	2487
	*	*	*	1	2.1	4.1	8.2
mit71	4207	1227	602	297	146	81	45
	1	3.4	7.0	14.2	28.9	52	93.5

Table 4: Large scale parallelization: secs/speedups, *= *Tsubame2* time limit exceeded

days longer than *cddr+*. *ppl_lcdd* ran for 76 days without giving any output and had to be terminated due to a power transfer. With only 4 cores both *mplrs* and *plrs* were able to outperform *cddr+*. With 1200 cores this problem was solved by *mplrs* in about 42 minutes, nearly 600 times faster than *cddr+*. We observe near linear speedup between 144 and 1200 cores. Solving in the 'reverse' direction is useful for checking the accuracy of a solution, and is usually extremely time consuming. For example, converting the V-representation of cp6 to an H-representation takes less than 2 seconds using any of the three single core codes.

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