

Comparative computational results for some vertex and facet enumeration codes

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Abstract

We report some computational results comparing parallel and sequential codes for vertex/facet enumeration problems for convex polyhedra. The problems chosen span the range from simple to highly degenerate polytopes. We tested one code (*lrs*) based on pivoting and four codes (*cddr+*, *ppl*, *normaliz*, *porta*) based on the double description method. *normaliz* employs parallelization as do the codes *plrs* and *mplrs* which are based on *lrs*. We tested these codes using various hardware configurations with up to 1200 cores. Major speedups were obtained by parallelization, particularly by the code *mplrs* which uses MPI and can operate on clusters of machines.

1 Background and polytopes tested

A convex polyhedron P can be represented by either a list of vertices and extreme rays, called a V-representation, or a list of its facet defining inequalities, called an H-representation. The vertex enumeration problem is to convert an H-representation to a V-representation. The computationally equivalent facet enumeration problem performs the reverse transformation. For further background see G. Ziegler [12].

In this note we consider only polytopes (bounded polyhedra) so extreme rays will not be required. Furthermore, for technical simplicity in this description, we assume that all polytopes are full dimensional. Neither condition is required for the algorithms tested and in fact some of our test problems are not full dimensional. The input for either problem is represented by an m by n matrix. For the vertex enumeration problem this is a list of m inequalities in $n - 1$ variables whose intersection define P . For a facet enumeration problem it is a list of the vertices of P each beginning with a 1 in column one¹. So in either case, under our assumption, the dimension of P is $n - 1$.

One of the features of this type of enumeration problem is that the output size varies widely for given input parameters m and n . This is shown explicitly by McMullen's Upper Bound Theorem (see, e.g., [12]) which is tight. It states that for a vertex enumeration problem with parameters m, n we have:

$$|V| \leq \binom{m - \lfloor \frac{n}{2} \rfloor}{m - n + 1} + \binom{m - \lfloor \frac{n+1}{2} \rfloor}{m - n + 1} \quad (1)$$

where $|V|$ is the number of vertices that are output. For a facet enumeration problem, by polarity of polytopes, the same inequality holds if we replace $|V|$ by $|F|$, the number of facets. By inverting the formula we can get lower bounds on the output size.

A class of polytopes for which the bound (1) is tight are the *cyclic polytopes* which are usually given as a V-representation consisting of m points on the $(n - 1)$ -dimensional moment curve. So, for example, a cyclic polytope with $m = 40$ and $n = 21$ has $|V| = 40$ vertices in dimension 20 and $|F| = 40,060,020$ facets. This implies that if we started with its H-representation, i.e., $m = 40,060,020$ and $n = 21$, then

¹Extreme rays would be indicated by a zero in column one.

the output would consist of only 40 vertices! Problems of this second type are called *highly degenerate* since each vertex may be described by many different combinations of facets. This contrasts with a *simple* polytope where each vertex is given by the intersection of exactly $n - 1$ facets. Dually a *simplicial* polytope is one where each facet contains precisely $n - 1$ vertices. Cyclic polytopes are simplicial.

The polytopes we tested are described in Table 1 and range from simple polyhedra to highly degenerate polyhedra. This table includes the results of an *lrs* run on each polytope as *lrs* gives the number of cobases in a symbolic perturbation of the polytope, showing how degenerate the polytope is. The corresponding input files are contained in the *lrslib-062* distribution [2] in subdirectory *lrslib-062/ine/test-062*. Note that the input sizes are small, roughly comparable and, except for *cp6*, much smaller than the output sizes. Five of the problems were previously used in [5]:

- *c30-15*, *c40-21*: cyclic polytopes described above. These have very large integer coefficients, the longest having 23 digits for *c30-15* and 33 digits for *c40-21*.
- *mit*: a configuration polytope used in materials science, created by G. Garbulsky [7]. The inequality coefficients are mostly integers in the range ± 100 with a few larger values.
- *perm10*: the permutahedron for permutations of length $N = 10$, whose vertices are the $10!$ permutations of $(1, 2, 3, \dots, 10)$. It is a 9-dimensional simple polytope. More generally, for permutations of length N , this polytope is described by $2^N - 2$ facets and one equation and has $N!$ vertices. The variables all have coefficients 0 or 1.
- *bv7*: an extended formulation of the permutahedron based on the Birkhoff-Von Neumann polytope. It is described by N^2 inequalities and $3N - 1$ equations in $N^2 + N$ variables and also has $N!$ vertices. The inequalities are all 0, ± 1 valued and the equations have single digit integers. The input matrix is very sparse and the polytope is highly degenerate.

The new problems are:

- *fq48-19*: related to the travelling salesman problem for $N = 5$, created by F. Quondam (private communication). The coefficients are all 0, ± 1 valued and it is moderately degenerate.
- *mit71*: a correlation polytope related to problem *mit*, created by G. Garbulsky [7]. The coefficients are similar to *mit* and it is moderately degenerate.
- *zfw91*: 0, ± 1 polytope based on a sensor network that is extremely degenerate and has large output size, created by Z.F. Wang [10]. There are three non-zeroes per row.
- *cp6*: the cut polytope for $N = 6$ solved in the ‘reverse’ direction: from an H-representation to a V-representation. The output consists of the 32 cut vectors of K_6 . It is extremely degenerate, approaching the lower bound of 19 vertices implied by (1) for these parameters. The coefficients of the variables are 0, ± 1 , ± 2 .

Name	Input				Output		<i>lrs</i>		
	H/V	m	n	size	V/H	size	bases	depth	secs
<i>bv7</i>	H	69	57	8.1K	5040	867K	84707280	17	8300
<i>c30-15</i>	V	30	16	4.7K	341088	73.8M	319770	14	39
<i>c40-20</i>	V	40	21	12K	40060020	15.6G	20030010	19	9445
<i>fq48-19</i>	H	48	19	1.3K	119184	8.7M	7843390	24	251
<i>mit71</i>	H	71	61	9.5K	3149579	1.1G	57613364	20	20688
<i>mit</i>	H	729	9	21K	4862	196K	1375608	101	496
<i>perm10</i>	H	1023	11	29K	3628800	127M	3628800	45	2230
<i>zfw91</i>	H	91	38	7.1K	2787415	205M	10819289888124 [†]		*
<i>cp6</i>	H	368	16	18K	32	1.6K	4844923002	153	1762156 [‡]

Table 1: Polytopes tested and *lrs* times(*mai20*): *=time > 604800 secs

[†]Computed by *mplrs1* v6.2 in 2144809 seconds using 289 cores (see Section 2).

[‡]Computed by *lrs* v6.0

2 Algorithms, implementations and machines used

There are basically two approaches to this problem: pivoting using reverse search [3] and the Fourier-Motzkin double description method, see [12]. The conventional wisdom is to use the double description method if the polytope is highly degenerate and use a pivoting method if it is simple or has low degeneracy (see e.g. [1], Section 3). Results below shed some doubt on the first part of this rule, especially when parallel processing is used. They do, however, confirm the second part of the rule. We tested five sequential codes, including four based on the double description method and one based on pivoting:

- *cddr+* (v. 0.77): Double description code developed by K. Fukuda [9].
- *normaliz* (v. 3.0.0): Hybrid parallel double description code developed by the Normaliz project [11].
- *porta* (v. 1.4.1): Double description code developed by T. Christof and A. Lobel [8].
- *ppl* (v. 1.1): Double description code developed by the Parma Polyhedral Library project [6].
- *lrs* (v. 6.2): C vertex enumeration code based on reverse search developed by D. Avis [2].

Of these five codes, *lrs* and *normaliz* offer parallelization. For *normaliz* this occurs automatically with the standard implementation if it is run on a shared memory multicore machine. The number of cores used can be controlled with the `-x` option, which we used extensively in our tests. For *lrs* two wrappers have been developed:

- *plrs* (v. 6.2): C++ wrapper for *lrs* using the Boost library, developed by G. Roumanis [5]. It runs on a single shared memory multicore machine.
- *mplrs* (v. 6.2): C wrapper for *lrs* using the MPI library, developed by the authors [4]. It runs on a network of multicore machines.

For *cp6* the *lrs* times in Tables 1–2 and the *plrs* times in Table 4 were obtained using v. 6.0 which has a smaller backtrack cache size than v. 6.2. Hence the *mplrs* speedups against *lrs* for *cp6* in Table 2 are probably somewhat larger than they would be against *lrs* v. 6.2. The *mplrs* times in Table 6 were obtained using v. 5.1b.

All of the above codes compute in exact integer arithmetic and with the exception of *porta*, are compiled with the GMP library for this purpose. However *normaliz* uses hybrid arithmetic, giving a very large speedup for certain inputs as described in the next section. In addition, *porta* can be run in either fixed or extended precision.

Finally, *lrs* is also available in a fixed precision 64-bit version, *lrs1*, which does no overflow checking. In general, this gives unpredictable results that need independent verification. In practice, for cases when there is no arithmetic overflow, *lrs1* runs about 4–6 times faster than *lrs* (see Computational Results on the *lrs* home page [2]). The parallel version of *lrs1*, *mplrs1*, was used to compute the number of cobases for *zfw91*, taking roughly 25 days on 289 cores.

The tests were performed using the following machines:

- *mai20*: 2x Xeon E5-2690 (10-core 3.0GHz), 20 cores, 128GB memory, 3TB hard drive
- *mai32abcd*: 4 nodes, each containing: 2x Opteron 6376 (16-core 2.3GHz), 32GB memory, 500GB hard drive (128 cores in total)
- *mai32ef*: 4x Opteron 6376 (16-core 2.3GHz), 64 cores, 256GB memory, 4TB hard drive
- *mai64*: 4x Opteron 6272 (16-core 2.1GHz), 64 cores, 64GB memory, 500GB hard drive
- *mai12*: 2x Xeon X5650 (6-core 2.66GHz), 12 cores, 24GB memory, 60GB hard drive
- *mai24*: 2x Opteron 6238 (12-core 2.6GHz), 24 cores, 16GB memory, 600GB RAID5 array
- *Tsubame2*: supercomputer located at Tokyo Institute of Technology

The first six machines total 312 cores and are located at Kyoto University. They were purchased between 2011-15 for a combined total of 3.9 million yen (\$33,200).

3 Computational results

Table 2 contains the results obtained by running the five sequential codes on the problems described in Table 1. The times for *lrs* shown in Table 1 are included for comparison. The time limit was one week (604,800 seconds) except for *cp6*. Programs *cddr+*, *lrs*, *ppl* were used with no parameters.

Name	<i>lrs</i>	<i>cddr+</i>	<i>ppl</i>	<i>normaliz</i>		<i>porta</i>	
	secs	secs	secs	secs(hybrid)	secs(GMP)	secs(64-bit)	secs(extended)
<i>bv7</i>	8300	*	578	122	1030	315	310
<i>c30-15</i>	39	2991	3040	**	**	**	**
<i>c40-20</i>	9445	*	*	**	**	**	**
<i>fq48-19</i>	252	437	1355	41	300	5103	4561
<i>mit71</i>	20688	*	260347	503564	364354	108993	107689
<i>mit</i>	496	368	40644	175	2174	**	47478
<i>perm10</i>	2230	*	*	1025	33240	*	*
<i>zfw91</i>	*	*	*	189763	*	31348	30787
<i>cp6</i>	1762156 [§]	1463829	>6570000	138162	1518785	**	>4925580

Table 2: Single processor times(*mai20*): *=time > 604800 secs **=abnormal termination

The program *normaliz* performs many additional functions, but was set to perform only vertex enumeration/facet enumeration for these tests. By default, it begins with 64-bit integer arithmetic and only switches to GMP arithmetic (used by all other programs except *porta*) in case of overflow. In this case, all work done with 64-bit arithmetic is discarded. For our test problems this happens on *c30-15*, *c40-20* and *mit71*, however the first two problems terminated abnormally after switching to GMP. Using the -B flag *normaliz* will do all computations using GMP arithmetic. We give times for the default hybrid arithmetic and also for GMP-only arithmetic. Note that *mit71* runs significantly faster with the -B flag reflecting the time wasted in 64-bit arithmetic mode.

As mentioned above, *porta* supports arithmetic using either 64-bit integers or its own extended precision arithmetic package. The program terminates if overflow occurs. We tested both options on each problem and found the extended precision option outperformed the 64-bit option in all cases.

It is hard to draw many general conclusions from the results in Table 2; especially since the four double description implementations behaved remarkably differently on most of the problems. This could be due to the fact that this method is highly sensitive to the insertion order of the input and the codes may be using different orderings. One clear result was that none of these codes could solve the cyclic polytope *c40-20* problem and struggled even on *c30-15*. We also observed that the double description codes use substantial memory, especially *normaliz*. In fact the machines with 32GB or less of memory were not able to solve either *mit71* or *cp6* using these codes, and even in single processor mode most of the 128GB available on *mai20* was required for some problems. Memory use by *lrs/plrs/mplrs* was negligible, making them good background processes. On the extremely degenerate problem *cp6*, *lrs* was in the middle of the pack, about 20% slower than *cddr+*, whereas *normaliz* was nearly 13 times faster than *lrs*. On this problem, neither *ppl* nor *porta* was able to produce any output in the time allotted (76 and 57 days respectively). Only *porta* and *normaliz* could effectively solve the sparse $0, \pm 1$ polytope *zfw91*. A 289-core run with *mplrs1* was approximately 70 times slower than *porta* and 11 times slower than *normaliz*. Note that with about 151 million cobases/vertex *cp6* is far more degenerate than *zfw91*, which has about 4 million cobases/vertex.

To put the above results in perspective, we recall that the problem *mit* was a big challenge in the early 1990s. At that time, early versions of both *cddr+* and *lrs* took over a month to solve this problem. Combined hardware and software improvements over the years give speedups of over 5000 times; both codes now complete the job in less than 10 minutes. We will see that parallelization of *lrs* can lead to further dramatic reductions in running time: on our 312-core cluster the problem now requires only 12 seconds.

We move now to the three parallel codes. For *mplrs* and *plrs* we used the default settings (see User's guide [2] for details):

- *plrs*: -id 4

[§]Computed by *lrs* v6.0

- *mplrs*: -id 2, -lmin 3 -maxc 50 -scale 100 -maxbuf 500

For *normaliz* we used the default settings which imply that hybrid arithmetic is used. Table 3 contains results for low scale parallelization and all problems were run on the single workstation *mai20*. With 4 cores available, *plrs* usually outperforms *mplrs*, they give similar performances with 8 cores, and *mplrs* is usually faster with 12 or more cores. With 16 cores *mplrs* gave quite consistent speedups, in the range 10-12.3. On the problems it could solve, the speedups obtained by *normaliz* show a much higher variance, in the range 0.93-15.7.

Name	4 cores secs/speedup			8 cores secs/speedup			12 cores secs/speedup			16 cores secs/speedup		
	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>
<i>bv7</i>	5219 1.6	2399 3.5	43 1.4	1739 4.8	1213 6.9	23 2.6	1045 8.0	818 10.2	17 3.7	747 11.1	624 13.3	13 4.8
<i>c30-15</i>	28 1.4	17 2.4	** -	9 4.4	11 3.6	** -	6 6.7	9 4.4	** -	4 10	10 4	** -
<i>c40-20</i>	5979 1.6	3628 2.6	** -	2023 4.7	2564 3.7	** -	1219 7.7	2237 4.2	** -	873 10.8	2066 4.6	** -
<i>fq48-19</i>	146 1.7	99 2.6	18 2.3	49 5.2	52 4.7	13 3.2	30 8.3	36 7.0	12 3.4	21 12.0	29 12.7	12 3.4
<i>mit71</i>	11386 1.8	6479 3.2	107482 4.7	3983 5.2	3320 6.2	65507 7.7	2390 8.6	2254 9.2	50910 9.9	1709 12.1	1724 12.0	42916 11.7
<i>mit</i>	293 1.7	152 3.3	70 2.5	99 5.0	89 5.6	42 4.2	61 8.1	68 7.3	33 5.3	44 11.3	57 8.7	29 6
<i>perm10</i>	1422 1.6	709 3.2	1085 .94	481 4.7	445 5.0	960 1.1	292 7.7	367 6.1	1090 .94	215 10.4	320 7.0	1093 .93
<i>zfw91</i>	* -	* -	46741 4.1	* -	* -	23885 7.9	* -	* -	15975 11.9	* -	* -	12110 15.7
<i>cp6</i>	968550 1.8	486667 3.6	42774 3.2	331235 5.3	268066 6.6	23493 5.9	199501 8.8	201792 8.7	18585 7.4	143006 12.3	169352 10.4	16980 8.1

Table 3: Small scale parallelization(*mai20*): *=time > 604800 secs, **=abnormal termination

Table 4 contains results for medium scale parallelization on the 64-core shared memory machine *mai64*. Note that these processors are considerably slower than *mai20* on a per-core basis. We used 8,16,32,64 cores and speedups are measured by comparing with the running time on 8 cores. With 64 cores, *mplrs* was the clear winner over *plrs* with speedups ranging from 4.3 to 7.2. *plrs* showed little improvement after 32 cores and *normaliz* again had very large variance.

Name	8 cores secs/speedup vs 8 cores			16 cores secs/speedup vs 8 cores			32 cores secs/speedup vs 8 cores			64 cores secs/speedup vs 8 cores		
	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>	<i>mplrs</i>	<i>plrs</i>	<i>normaliz</i>
<i>bv7</i>	3238 1	2255 1	60 1	1478 2.2	1212 1.9	39 1.5	1206 2.7	726 3.1	29 2.1	515 6.3	506 4.4	21 2.9
<i>c30-15</i>	17 1	22 1	** -	9 1.9	20 1.1	** -	5 3.4	22 1.0	** -	4 4.3	21 1.0	** -
<i>c40-20</i>	3882 1	4694 1	** -	1876 2.1	4163 1.1	** -	1141 3.4	4192 1.1	** -	717 5.4	4086 1.1	** -
<i>fq48-19</i>	89 1	95 1	28 1	42 2.1	57 1.7	24 1.2	23 3.9	39 2.4	22 1.3	14 6.4	31 3.1	23 1.2
<i>mit71</i>	7395 1	6218 1	115088 1	3401 2.2	3441 1.8	77436 1.5	1900 3.9	2130 2.9	60694 1.9	1251 5.9	1640 3.8	51594 2.2
<i>mit</i>	195 1	175 1	111 1	93 2.1	123 1.4	83 1.3	53 3.7	120 1.5	75 1.5	42 4.6	124 1.4	82 1.4
<i>perm10</i>	909 1	841 1	1951 1	432 2.1	617 1.4	1870 1.1	253 3.6	569 1.5	1840 1.1	171 5.3	573 1.5	1930 1
<i>zfw91</i>	* -	* -	42409 1	* -	* -	24822 1.7	* -	* -	14452 2.9	* -	* -	7332 5.8
<i>cp6</i> ⁵	727771 1	565915 1	38621 1	326214 2.2	377857 1.5	23773 1.6	171194 4.3	298408 1.9	17468 2.2	100676 7.2	229713 2.5	15480 2.5

Table 4: Medium scale parallelization (*mai64*): *=time > 604800 secs, **=abnormal termination

Table 5 contains results for medium scale parallelization on a 312-core cluster of computers. Only *mplrs* is able to use all cores in this heterogeneous environment. The machines were scheduled in the

⁵*plrs* times computed using v6.0

order given at the end of Section 2 (excluding *Tsubame2*). Due to the heterogeneous selection of machines we do not present speedups in this table. For example, we observed that *mai20* is substantially faster than the other machines – more than would be expected by simply comparing clock speeds and number of cores. It was more than twice as fast as *mai12* on *c40-20*. Jobs completing in under a minute do not profit much, if at all, as extra cores are added. However, the longer running jobs show continuous improvement. Excluding *zfw91*, *lrs* required about 3 weeks on the fastest machine (*mai20*) to complete the other 9 problems. Using the 312-core cluster this time is reduced to 4 hours and 40 minutes. These total times are dominated by *cp6*. Excluding this problem as well, the *lrs* total running time of 12 hours 13 minutes is improved to roughly 8 minutes using the cluster.

Name	<i>mplrs</i>					
	16 cores secs	32 cores secs	64 cores secs	128 cores secs	256 cores secs	312 cores secs
<i>bv7</i>	747	389	262	179	101	88
<i>c30-15</i>	4	3	2	3	2	2
<i>c40-20</i>	873	528	328	218	133	121
<i>fq48-19</i>	21	11	7	5	4	5
<i>mit71</i>	1709	956	625	421	228	199
<i>mit</i>	44	26	21	23	13	12
<i>perm10</i>	215	118	89	75	53	55
<i>cp6</i>	143006	75712	50225	33684	18657	16280

Table 5: Medium scale parallelization(cluster)

Table 6 shows results for large scale parallelization obtained by Kazuki Yoshizoe using *mplrs* v. 6.0 on the *Tsubame2* supercomputer at the Tokyo Institute of Technology. He ran tests using problems *mit71* and *cp6* and observed near linear speedup between 12 and 1200 cores for both problems⁶. With 1200 cores *mplrs* solved *cp6* in about 42 minutes, nearly 600 times faster than *cddr+*, 55 times faster than *normaliz* in single processor mode and over 6 times faster than *normaliz* running on 64 cores, the largest shared memory machine available to us.

Name	<i>mplrs</i> (v. 5.1b)						
	12 cores	36 cores	72 cores	144 cores	300 cores	600 cores	1200 cores
<i>cp6</i>	283403(<i>mai12</i>)	*	*	20383	9782	4913	2487
	1	-	-	14	29	58	114
<i>mit71</i>	4207	1227	602	297	146	81	45
	1	3.4	7.0	14	29	52	94

Table 6: Large scale parallelization: secs/speedups, *=*Tsubame2* time limit exceeded

4 Conclusions

These results show that the difficulty of solving vertex/facet enumeration problems varies enormously, even for inputs of roughly the same size. Any given problem may be tractable or intractable depending on the method used to solve it. General rules are dangerous and likely to be contradicted by further examples, but we hazard two: learn about your polytope and use multicore hardware.

4.1 Learn something about your polytope

Unfortunately not much can be learned by simply inspecting the input file. Many 0/1 input files are highly degenerate, but not all: *perm10*, for example, is a simple polytope. Fortunately the degeneracy of a polytope can be checked by doing a partial run of *lrs* for a few minutes, stopping after a certain number of bases have been computed. As seen from Table 1, the ratio of bases computed to V/H output gives a good estimate of degeneracy of the problem. It will also give an indication as to whether the

⁶*cp6* benchmark was taken with *mai12* which has a similar processor (Xeon X5650) to those we used on *Tsubame2* (Xeon X5670).

output is binary (*cp6*), consists of small integers (*bv7*, *perm10*, *zfw91*), huge integers (*c30-15*, *c40-20*) or rationals (*fq48-19*, *mit*, *mit71*). *lrs* also has an estimate feature that gives an unbiased estimate of the output size, number of bases and total *lrs* running time. These estimates have high variance but do give some indication of the tractability of the problem.

For problems with low degeneracy or very large output sizes pivoting methods such as the *lrs* family may be the only tractable approach. For extremely degenerate problems with binary or small integer output it is not so clear, as can be seen by comparing the results obtained for *cp6* and *zfw91*.

4.2 Use multicore hardware

Comparing Table 2 with the remaining tables clearly indicates the necessity of using parallel processing for hard vertex/facet enumeration problems: even just 16 cores gives an order of magnitude improvement. A supercomputer on the scale of *Tsubame2* may seem out of reach for most researchers. However, at current prices, a 1200-core cluster could be built for roughly \$100,000 and would be considerably cheaper with used hardware. This price will certainly fall substantially in the near future making this amount of computing power readily available to more researchers. The problem will not be the availability of the hardware but the availability of software that can make effective use of it.

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