INFORMATION AND COMPUTATION 92, 253–284 (1991)

One-Sided Error Probabilistic Inductive Inference and Reliable Frequency Identification

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For EX- and BC-type identification, one-sided error probabilistic inference and reliable frequency identification on sets of functions are introduced. In particular, we relate the one to the other and characterize one-sided error probabilistic inference to exactly coincide with reliable frequency identification, on any set $\mathfrak{M}$. Moreover, we show that reliable EX and BC-frequency inference forms a new discrete hierarchy having the breakpoints 1, 1/2, 1/3, ...

1. INTRODUCTION

Inductive inference has its historical origins in the philosophy of science. Within the last two decades it has attracted much attention from computer scientists. The theory of inductive inference can be considered as a form of machine learning with potential applications to artificial intelligence (cf. Osherson *et al.*, 1986). Nowadays inductive inference is a well-developed mathematical theory which has been the subject of collections of papers (cf. Barzdin, Ed., 1974, 1975, 1977) and of several excellent survey papers (cf. Angluin and Smith, 1983, 1986; Daley, 1986; Klette and Wiehagen, 1980). Part of the following work was suggested by an open problem presented in Daley (1986).

As in previous studies, we deal with the synthesis of programs for recursive functions. An inductive inference machine (abbr. IIM) is a recursive device (deterministic, probabilistic, or pluralistic) which, when fed more

* The results were obtained during the author's visit to the computing center of the Latvian State University.

0890-5401/91 $3.00

and more ordered pairs from the graph of some function, outputs more and more hypotheses, i.e., programs. There are many possible requirements on the sequence of all actually created programs. Considering deterministic and probabilistic IIM we shall study explanatory (EX, $\mathrm{EX}_{\mathrm{prob}}$) inference, where the sequence of programs has to converge to a single program correctly computing the function to be identified, as well as behaviorally correct (BC, $\mathrm{BC}_{\mathrm{prob}}$) inference; i.e., all but finitely many programs have to satisfy the relevant correctness criterion. Furthermore, the correctness criteria range from absolute correctness to finite error tolerance. On the other hand, investigating pluralistic IIMs we shall distinguish between the following two cases: First, the sequence of programs is required to contain a *particular* correct hypothesis with a certain *frequency* ($\mathrm{EX}_{\mathrm{freq}}$). In the second case the sequence of programs is required to contain with a certain *frequency* correct but possibly *distinct* programs ($\mathrm{BC}_{\mathrm{freq}}$). The reader is encouraged to consult Case and Smith (1983), Pitt (1989), and Podnieks (1974, 1975) for further information.

Moreover, we generalize the reliability notion originally introduced by Blum and Blum (1975) and Minicozzi (1976) to all these modes of identification. For EX-type inference, an IIM works reliably on a certain set $\mathfrak{M}$ of functions (e.g., the total functions or the recursive functions) if for every function from $\mathfrak{M}$ the sequence of created programs does not converge to an incorrect solution. Hence the IIM itself recognizes whether its last hypothesis may be or may not be correct. In the latter case it performs a mind change; i.e., it outputs a program different from the previous one. Thereby the IIM M implicitly transmits an error message to the outside word. If M identifies some function from $\mathfrak{M}$ then its sequence of error messages is finite, otherwise it is infinite. Thus our generalization works as follows: Instead of outputting programs, now the IIM M is required to output ordered pairs (i_n, b), where the i_n are the programs and $b \in \{0, 1\}$. If $b = 0$, we interpret $(i_n, 0)$ as an error message. In other words, $b = 1$ indicates that M trusts in its current hypothesis. If M does not identify some function from $\mathfrak{M}$ then it again produces an infinite sequence of error messages. Otherwise, the output sequence contains only finitely many error messages and among all created programs there are, with a certain frequency, correct ones.

Transmitting this approach to probabilistic IIMs we get that on some function from $\mathfrak{M}$ all possible computations yield either an infinite sequence of error messages or a finite one, independently of the sequence of coin-flips. Furthermore, in the latter case, with a certain *probability*, there must occur sequences of programs satisfying the particular identification criterion. Hence, all uncertainty lies in the domain of identification. Consequently we can interpret this type of probabilistic identification as *one-sided error probabilistic inference.*

In the present paper, first we extent Pitt's (1984, 1989) unification results in characterizing one-sided error probabilistic inference to coincide with reliable frequency identification. Second we prove that the introduced reliability notion ensures the useful properties known for the ordinary case, i.e., closure under union and finite invariance (cf. Minicozzi, 1976). Third we investigate the power of reliable EX-type and BC-type frequency inference in comparing them with ordinary frequency identification. Thereby we obtain the strongest possible result, i.e., we show that there are classes that are reliable EX-identifiable on the set of all total functions with frequency $1/(k+1)$ not contained in $BC_{\text{freq}}(1/k)$, for all numbers k. This directly yields *four infinite hierarchies*. Finally we discuss open problems.

A picture showing the relationships between all the concepts of identification studied in the present paper is given at the end of Section 3.

2. Basic Definitions and Notations

Unspecified notations follow Rogers (1967). $\mathbb{N} = \{0, 1, 2, ...,\}$ denotes the set of all natural numbers. The classes of all partial recursive and recursive functions of n variables over $\mathbb{N}$ are denoted by $\mathbb{P}^n$ and $\mathbb{R}^n$, respectively. For $n = 1$ we omit the upper index. The classes of all partial and total functions over $\mathbb{N}$ are denoted by $\mathbb{PF}$ and $\mathbb{TF}$, respectively. Let $f \in \mathbb{PF}$; then we set $\operatorname{Arg} f = \{x/f(x) \text{ is defined}\}$ and $\operatorname{Val} f = \{f(x)/f(x) \text{ is defined}\}$. For $n \in \mathbb{N}$, we denote by $\mathbb{TF}_n$ and $\mathbb{R}_n$ the classes of all functions $f \in \mathbb{PF}$ and $f \in \mathbb{P}$, respectively, for which $\operatorname{card}(\mathbb{N}\text{-}\operatorname{Arg} f) \leqslant n$. The classes of all functions $f \in \mathbb{P}$ and $f \in \mathbb{PF}$ with cofinite domains are denoted by $\mathbb{R}_*$ and $\mathbb{TF}_*$, respectively. Let f, $g \in \mathbb{TF}_*$ and $n \in \mathbb{N}$; we write $f(n) \leqslant g(n)$ if both $f(n)$ and $g(n)$ are defined and $f(n)$ is not greater than $g(n)$. Furthermore, for f, $g \in \mathbb{TF}_*$ and $n \in \mathbb{N}$ we write $f =_n g$ and $f =_* g$ iff $\operatorname{card}(\{x/f(x) \neq g(x)\}) \leqslant n$ and $\operatorname{card}(\{x/f(x) \neq g(x)\}) < \infty$, respectively. Let $f \subseteq g$ iff $\{(x, f(x))/x \in \operatorname{Arg} f\} \subseteq \{(x, g(x))/x \in \operatorname{Arg} g\}$.

By $\varphi_0, \varphi_1, \varphi_2, ...$, we denote a fixed acceptable programming system of all (and only all) the partial recursive functions, and by $\Phi_0, \Phi_1, \Phi_2, ...$, an associated computational complexity measure (cf. Machtey and Young, 1978). If $f \in \mathbb{P}$ and $i \in \mathbb{N}$ are such that $\varphi_i = f$ then i is called a program for f. If $\varphi_i(x)$ is defined (written: $\varphi_i(x)\downarrow$), we also say that $\varphi_i(x)$ converges; otherwise $\varphi_i(x)$ diverges (written $\varphi_i(x)\uparrow$).

Sometimes it will be suitable to identify a recursive function with the sequence of its values; e.g., 0^i10^∞ denotes the function f for which $f(i) = 1$ and $f(x) = 0$ for all $x \neq i$. Using a fixed effective encoding $\langle \cdots \rangle$ of all finite sequences of natural numbers onto $\mathbb{N}$ we write f^n instead of $\langle (f(0), ..., f(n)) \rangle$, for any $n \in \mathbb{N}$, $f \in \mathbb{PF}$, where $f(x)$ is defined for all $x \leqslant n$.

Proper set inclusion is denoted by $\subset$ in distinction from $\subseteq$; and by $\#$ we denote incomparability of sets.

A sequence $(j_n)_{n \in \mathbb{N}}$ of natural numbers is said to be convergent to a number j iff $j_n = j$ for almost all n.

Now we define several concepts of identification. In the sequel we deal only with the inference of everywhere defined functions, since this suffices to get the desired results. Unless otherwise stated, an IIM M is just a partial recursive function. Suppose an IIM M is given the graph of some function $f \in \mathbb{TF}$ as input. We may suppose without loss of generality that f is given in its natural order $(f(0), f(1), ...,)$ to M (cf. Blum and Blum, 1975).

DEFINITION 1. Let $a \in \mathbb{N} \cup \{*\}$, and let $f \in \mathbb{TF}$. An IIM M EX^a-identifies f iff $M(f^n)$ is defined for any $n \in \mathbb{N}$, and the sequence $(M(f^n))_{n \in \mathbb{N}}$ converges to a number i such that $\varphi_i =_a f$.

If M does EX^a-identify f, we write $f \in \mathrm{EX}^a(M)$. The collection of EX^a-inferrible sets is denoted by EX^a; formally, $\mathrm{EX}^a = \{U / \exists M\ [U \subseteq \mathrm{EX}^a(M)]\}$. For $a = 0$ we omit the upper index.

EX-identification was originally introduced by Gold (1965) (so-called identification in the limit), whereas the $a = *$ case was studied first by Blum and Blum (1975). Furthermore, Case and Smith (1983) have investigated EX^a-identification for all $a \in \mathbb{N} \cup \{*\}$, obtaining the hierarchy

$$\mathrm{EX} \subset \mathrm{EX}^1 \subset \cdots \subset \bigcup_{a \in \mathbb{N}} \mathrm{EX}^a \subset \mathrm{EX}^*.$$

Interesting results concerning the power of EX^a-identification can also be found in Chen (1982).

DEFINITION 2. Let $\mathfrak{M} \subseteq \mathbb{TF}$, and let $a \in \mathbb{N} \cup \{*\}$. An IIM $M \in \mathbb{R}$ works EX^a-reliably on the set $\mathfrak{M}$ iff for every function $f \in \mathfrak{M}$ *either* the sequence $(M(f^n))_{n \in \mathbb{N}}$ converges to a number i such that $\varphi_i =_a f$ *or* it diverges.

DEFINITION 3. Let $a \in \mathbb{N} \cup \{*\}$, and let $\mathfrak{M} \subseteq \mathbb{TF}$. An IIM M *reliably* EX^a-identifies $f \in \mathbb{TF}$ on the set $\mathfrak{M}$ iff M works EX^a-reliably on the set $\mathfrak{M}$ and M EX^a-identifies f.

If M does reliably EX^a-identify f on the set $\mathfrak{M}$ we write $f \in \mathfrak{M}\text{-}\mathrm{REX}^a(M)$. The collection of reliably on $\mathfrak{M}$ EX^a-identifiable sets is denoted by $\mathfrak{M}\text{-}\mathrm{REX}^a$. Again, for $a = 0$ we omit the upper index.

Reliably working IIMs were originally introduced and studied by Minicozzi (1976) and Blum and Blum (1975), in case $a = 0$, $*$. In Kinber and

Zeugmann (1985) the collections $\mathbb{TF}$-REX^a and $\mathbb{R}$-REX^a have been considered and the following hierarchy was pointed out:

$$\begin{array}{ccccccc}
\mathbb{TF}\text{-REX} \subset & \mathbb{TF}\text{-REX}^1 \subset & \cdots \subset & \bigcup_{a \in \mathbb{N}} & \mathbb{TF}\text{-REX}^a \subset & \mathbb{TF}\text{-REX}^* \\
\cap & \cap & & & \cap & \cap \\
\mathbb{R}\text{-REX} \subset & \mathbb{R}\text{-REX}^1 \subset & \cdots \subset & \bigcup_{a \in \mathbb{N}} & \mathbb{R}\text{-REX}^a \subset & \mathbb{R}\text{-REX}^*
\end{array}$$

Next we consider behaviorally correct inference which has been introduced by Barzdin (1974) and which has been studied intensively in Case and Smith (1983).

Definition 4. Let $a \in \mathbb{N} \cup \{*\}$, and let $f \in \mathbb{TF}$. An IIM M BC^a-identifies f iff $M(f^n)$ is defined for all $n \in \mathbb{N}$, and $\varphi_{M(f^n)} =_a f$, for almost all n.

We write $f \in \text{BC}^a(M)$, if M does BC^a-identify f and set $\text{BC}^a = \{U / \exists M[U \subseteq \text{BC}^a(M)]\}$.

Case and Smith (1983) proved that

$$\text{EX}^* \subset \text{BC} \subset \text{BC}^1 \subset \cdots \subset \bigcup_{a \in \mathbb{N}} \text{BC}^a \subset \text{BC}^*,$$

where the first inclusion was first shown in Barzdin (1974). Moreover, Leo Harrington discovered that $\mathbb{R} \in \text{BC}^*$ (cf. Case and Smith, 1983). The reader is also encouraged to consult Chen (1981) for more results concerning BC^a-identification (e.g., the complexity of the synthesized programs).

Now we introduce reliable BC^a-identification. As already explained in the Introduction, we now require the IIM to output ordered pairs (i_n, b), where $b \in \{0, 1\}$, instead of only outputting programs. If $b = 0$, then (i_n, b) is said to be an error message.

Definition 5. Let $\mathfrak{M} \subseteq \mathbb{TF}$, and let $a \in \mathbb{N} \cup \{*\}$. An IIM $M \in \mathbb{R}$ works BC^a-reliably on the set $\mathfrak{M}$ iff for every function $f \in \mathfrak{M}$ *either* the output sequence (i_n, b_n) satisfies $(i_n, b_n) = (i_n, 1)$ and $\varphi_{i_n} =_a f$, for almost all n, *or* there are infinitely many n such that $(i_n, b_n) = (i_n, 0)$.

In other words, either M does BC^a-identify f or it produces infinitely many error messages. Note that we *do not require* $(i_n, 1)$ to imply $\varphi_{i_n} =_a f$.

Definition 6. Let $a \in \mathbb{N} \cup \{*\}$, and let $\mathfrak{M} \subseteq \mathbb{TF}$. An IIM M reliably BC^a-identifies $f \in \mathbb{TF}$ on the set $\mathfrak{M}$ iff M works BC^a-reliably on $\mathfrak{M}$ and M does BC^a-identify f.

If M does reliably BC^a-identify f on $\mathfrak{M}$ we write $f \in \mathfrak{M}\text{-}RBC^a(M)$ and we denote by $\mathfrak{M}\text{-}RBC^a$ the collection of reliably BC^a-inferrible sets on $\mathfrak{M}$.

Please note that it is not hard at all to encode the error messages into the current hypotheses; e.g., we may force the IIM to output on f^n instead of $(i_n, 0)$ a program j_n such that $\varphi_{j_n}(x)\downarrow$ and $\varphi_{j_n}(x) \neq f(x)$, for all $x \leqslant n$. On the other hand, our formalization is technically much more convenient.

Next we deal with frequency identification due to Podnieks (1974).

DEFINITION 7. Let $0 < p \leqslant 1$, and let $f \in \mathbb{TF}$. An IIM M BC-identifies f with frequency p iff $M(f^n)$ is defined for all $n \in \mathbb{N}$ and

$$\liminf_{k \to \infty} \frac{\text{card}(\{n/\varphi_{M(f^n)} = f, 0 \leqslant n \leqslant k\})}{k} \geqslant p.$$

Then we set $BC_{\text{freq}}(p) = \{U/\exists M\ [M \text{ identifies every } f \in U \text{ with frequency } p]\}$.

Podnieks (1974, 1975) shows that $BC_{\text{freq}}(1/(n+1)) \subset BC_{\text{freq}}(1/(n+2))$, for any $n \in \mathbb{N}$. Moreover, he points out that the $BC_{\text{freq}}(p)$ hierarchy is discrete; i.e., given any p with $1/n \geqslant p > 1/(n+1)$, then $BC_{\text{freq}}(p) = BC_{\text{freq}}(1/n)$. Pitt (1989) has introduced what is essentially the EX version of Podnieks' BC-frequency identification, and he has proved that the analogous theorems are true.

DEFINITION 8. Let $0 < p \leqslant 1$, and let $f \in \mathbb{TF}$ An IIM M EX-identifies f with frequency p iff $M(f^n)$ is defined for all $n \in \mathbb{N}$, and there is a *particular* program i such that

$$\liminf_{k \to \infty} \frac{\text{card}(\{n/M(f^n) = i, 0 \leqslant n \leqslant k\})}{k} \geqslant p$$

and $\varphi_i = f$.

By $EX_{\text{freq}}(p)$ we denote the collection of all function classes U that are EX-identifiable with frequency p.

In order to introduce reliable frequency identification, the IIMs are again required to output pairs (i_n, b_n). Furthermore, looking at EX-frequency inference we require the particular program i computing the considered function to occur at least with frequency p. Moreover, the created output sequence is only allowed to contain finitely many error messages; i.e., the program i is at almost all places accompanied by $b = 1$.

DEFINITION 9. Let $\mathfrak{M} \subseteq \mathbb{TF}$, and let $0 < p \leqslant 1$. An IIM $M \in \mathbb{R}$ reliably BC-identifies (EX-identifies) $f \in \mathbb{R}$ on the set $\mathfrak{M}$ with frequency p iff

(1) for all $g \in \mathfrak{M}$ *either* the output sequence $(M(g^n))_{n \in \mathbb{N}}$ contains only finitely many error messages and M BC-identifies (EX-identifies) g with frequency p *or* in the sequence $(M(g^n))_{n \in \mathbb{N}}$ error messages occur infinitely often, and

(2) M BC-identifies (EX-identifies) f with frequency p.

If M does reliably BC-identify (EX-identify) f on the set $\mathfrak{M}$ with frequency p we write $f \in \mathfrak{M}\text{-RBC}_{\text{freq}}(p)(M)$ $(f \in \mathfrak{M}\text{-REX}_{\text{freq}}(p)(M))$. Furthermore we set $\mathfrak{M}\text{-RBC}_{\text{freq}}(p) = \{U/\exists M[U \subseteq \mathfrak{M}\text{-RBC}_{\text{freq}}(p)(M)]\}$, and analogously define $\mathfrak{M}\text{-REX}_{\text{freq}}(p)$.

Next we define team inference, which was originally introduced by Smith (1982). A team is a finite collection of IIMs. A team $(M_1, ..., M_n)$ successfully BC-infers (EX-infers) a set $U \subseteq \mathbb{R}$ if, for each $f \in U$, some team member M_i successfully BC-identifies (EX-identifies) f. Furthermore we set $\text{BC}_{\text{team}}(n) = \{U/\exists(M_1, ..., M_n)[U \subseteq \text{BC}(M_1, ..., M_n)]\}$, and declare analogously $\text{EX}_{\text{team}}(n)$.

Smith (1982) proves that $\text{BC}_{\text{team}}(n) \subset \text{BC}_{\text{team}}(n+1)$ as well as $\text{EX}_{\text{team}}(n) \subset \text{EX}_{\text{team}}(n+1)$ for any $n \geqslant 1$. The unifying results stating that $\text{BC}_{\text{freq}}(1/n) = \text{BC}_{\text{team}}(n)$ and $\text{EX}_{\text{freq}}(1/n) = \text{EX}_{\text{team}}(n)$ are due to Pitt (1989). Note that it is *not* meaningful to consider teams of BC^a-reliably working IIMs as well as of EX^a-reliably working IIMs, since the classes $\mathfrak{M}\text{-REX}^a$ (cf. Minicozzi, 1976) and $\mathfrak{M}\text{-RBC}^a$ (cf. Proposition 1) are closed under enumerable union. However, reliable frequency identification is a powerful tool combining the advantages of reliability with those of bounded non-determinism, as we shall show.

Finally, in this section we define one-sided error probabilistic inference. For the sake of intuition as well as for any mathematical background the reader is referred to Pitt (1984, 1989).

A probabilistic IIM P is simply a deterministic IIM which is allowed to flip a "t-sided coin." For any fixed sequence S of coin-flips P behaves like a deterministic IIM, which we denote by P^S. We again require P^S, for any sequence S of coin-flips, to output ordered pairs (i_n, b_n).

Definition 10. Let $\mathfrak{M} \subseteq \mathbb{TF}$, and let $0 < p \leqslant 1$. An one-sided error probabilistic IIM P on $\mathfrak{M}$ BC-identifies (EX-identifies) $f \in \mathbb{R}$ with probability p iff

(1) for all $g \in \mathfrak{M}$ and all sequences S of coin-flips the output sequence $(P^S(g^n))_{n \in \mathbb{N}}$ satisfies *either* (α) *or* (β).

(α) $(P^S(g^n))_{n \in \mathbb{N}}$ contains only finitely many error messages (independently of S) and the probability (taken over all sequences S) that P^S BC-identifies (EX-identifies) g is greater than or equal to p.

(β) in the sequence $(P^S(g^n))_{n \in \mathbb{N}}$ error messages occur infinitely often (again independently of S).

(2) The sequence $(P^S(f^n))_{n \in \mathbb{N}}$ fulfills (α).

If P does BC-identify f on $\mathfrak{M}$ with one-sided error probability p, we write $f \in \mathfrak{M}\text{-RBC}_{\text{prob}}(p)(P)$ and set $\mathfrak{M}\text{-RBC}_{\text{prob}}(p) = \{U/\exists P[U \subseteq \mathfrak{M}\text{-RBC}_{\text{prob}}(p)(P)]\}$. In an analogous way we define $\mathfrak{M}\text{-REX}_{\text{peob}}(p)(P)$ and $\mathfrak{M}\text{-REX}_{\text{peob}}(p)$.

In the sequel we shall study the unknown relationships between the above defined modes of inference.

3. Results

3.1. *One-Sided Error Probabilistic Inference and Reliable Frequency Identification*

In this section we extend Pitt's (1984) unification results to show that one-sided error probabilistic inference is exactly the same as reliable frequency identification. Moreover, as an immediate consequence from the theorem below one gets that, if there is any hierarchy for reliable frequency identification, then it must be a discrete one. As it turns out, Pitt's techniques of proof are powerful enough to show the desired results. Please note that if one is only interested in the discretness result concerning reliable frequency identification then shorter proofs can be obtained by applying Podnieks' (1975) techniques. However, since it is our goal to get more general theorems, we shall follow Pitt. Because Pitt's proofs work mutatis mutandis in our setting we shall describe only the minor changes which have to be made. Therefore, the reader is advised first to consult Chapters 2 and 3 of Pitt (1989).

Let P be any one-sided probabilistic IIM which is, without loss of generality, equipped with a two-sided coin, and realized by a Turing machine, and let $f \in \mathbb{TF}$. Then $\mathfrak{T}_{P,f}$ denotes the infinite complete binary computation tree which represents all the possible computations of P on input f.

Now we are ready to present the first theorems.

THEOREM 1. *Let* $n \in \mathbb{N}$, *and let* $\mathfrak{M} \subseteq \mathbb{TF}$. *Furthermore, let* $U \in \mathfrak{M}\text{-RBC}_{\text{prob}}(p)$ *with* $1 \geq p > 1/(n+1)$. *Then*

(1) *there is an* IIM M *reliably* BC-*identifying* U *on* $\mathfrak{M}$ *with frequency* $1/n$;

(2) *for any* $f \in U$ *the sequence* $((i_k, b_k)_{k \in \mathbb{N}}$ *of M's hypotheses on f has the property that there is some* $r \in \{0, ..., n-1\}$ *such that* $\varphi_{i_k} = f$, *for almost all k with* $k \equiv r \bmod n$.

Proof. The proof essentially coincides with Pitt's (1989) proof that $\mathrm{BC}_{\mathrm{prob}}(p) = \mathrm{BC}_{\mathrm{team}}(1/n)$. Hence we describe only the IIM which will reliably identify U on $\mathfrak{M}$ with frequency $1/n$, thereby satisfying (2).

Let $U \in \mathfrak{M}\text{-}\mathrm{RBC}_{\mathrm{prob}}(p)(P)$. The idea underlying our construction is quite similar to the idea used by Pitt to show that $\mathrm{BC}_{\mathrm{prob}}(p) = \mathrm{BC}_{\mathrm{team}}(n)$. We first ensure that for any $f \in \mathfrak{M}$ in which P produces infinitely many error messages (remember that this happens independently of the sequence of coin-flips) *each* team member also outputs infinitely many error messages. On the other hand, if P outputs only finitely many error messages on $f \in \mathfrak{M}$ (again independently of the coin-flips) then each team member behaves exactly as in Pitt's team.

This modification is achieved as follows:

We introduce an auxiliary parameter d having the initial value 1. The computation tree $\mathfrak{T}_{P,f}$ is now constructed stepwise; i.e., for $f \in \mathfrak{M}$ and $k = 1, 2, 3, ...$, the ith team member builds the finite computation tree $\mathfrak{T}_k$ just consisting of the first k levels of $\mathfrak{T}_{P,f}$. Then it first tests whether each path in $\mathfrak{T}_k$ contains at least d error messages. If the test is fulfilled, M_i outputs $(k, 0)$ and increments d, i.e., $d := d + 1$. Otherwise it tries to compute a new hypothesis by inspecting all nodes at level k. This actually requires the installation of a time regime in order to effectively execute instruction 4 in Pitt's definition of the team member M_i. If no new hypothesis can be computed within the given time then $M_i(f^k) = M_i(f^{k-1})$. For completness we define $M_i(f^0) := (0, 0)$.

Now it is obvious that M_i produces infinitely many error messages if P behaves thus. Suppose that P outputs only a finite number of error messages. Let d_0 be the least number of error messages taken over all paths in $\mathfrak{T}_{P,f}$. Consequently, for $d > d_0$ the test introduced above cannot be satisfied. Moreover, in accordance with the definition of $\mathfrak{M}\text{-}\mathrm{RBC}_{\mathrm{prob}}$ the IIM P must infer f. Hence M_i behaves in the limit as Pitt's ith team member does. Since in general not every team member identifies f, only those ones that infer f behave reliably on $\mathfrak{M}$, i.e., at least one team member. However, the desired IIM can now be defined. M simply outputs the hypotheses of the team members $M_0, ..., M_{n-1}$ in a rotating order. Suppose M_r identifies f. Then $\varphi_{i_k} = f$, for almost all k with $k \equiv r \bmod n$. Q.E.D.

THEOREM 2. *Let* $\mathfrak{M} \subseteq \mathbb{TF}$, $n \in \mathbb{N}$, *and let* $U \in \mathfrak{M}\text{-}\mathrm{RBC}_{\mathrm{freq}}(p)$ *with* $1 \geq p > 1/(n+1)$. *Then there is an* IIM *M reliably* BC-*identifying U with frequency* $1/n$. *Moreover, the output sequence* $((i_k, b_k))_{k \in \mathbb{N}}$ *of M's hypotheses has for* *any $f \in U$ the property that there is an $r \in \{0, ..., n-1\}$ such that $\varphi_{i_k} = f$ for almost all $k \equiv r$ mod n.*

Proof. The theorem is shown in the same way as Pitt (1989) proves his theorem stating $BC_{freq}(p) \subseteq BC_{team}(n)$, with the following minor differences.

Let $f \in \mathfrak{M}$, and let $k \in \mathbb{N}$. Suppose $U \in \mathfrak{M}\text{-}RBC_{freq}(p)(M')$. If $M'(f^k) = (i_k, 0)$ then each team member $M_0, ..., M_{n-1}$ straightforwardly outputs $(i_k, 0)$. Otherwise the team behaves exactly as Pitt's team does, but accompanies each guess with $b_k = 1$. Finally, the wanted IIM M again outputs the hypotheses of the team members in a rotating order. Q.E.D.

COROLLARY 3. *Let $\mathfrak{M} \subseteq \mathbb{TF}$. Then $\mathfrak{M}\text{-}RBC_{prob}(1/n) = \mathfrak{M}\text{-}RBC_{freq}(1/n)$, for any number $n \geq 1$.*

Proof. Due to Theorem 1 we have $\mathfrak{M}\text{-}RBC_{prob}(1/n) \subseteq \mathfrak{M}\text{-}RBC_{freq}(1/n)$. In order to prove the other inclusion, first we apply Theorem 2. Without loss of generality we may assume that the desired probabilistic IIM P is equipped with an n-sided coin. P flips the coin once and obtains, with probability $1/n$, a particular value $r \in \{0, ..., n-1\}$. Then it outputs only the guesses (i_k, b_k) with $k \equiv r$ mod n.

The rest is obvious. Q.E.D.

Finally, in this section we show that Theorems 1 and 2 as well as Corollary 3 remain valid if reliable BC-frequency identification is replaced by reliable EX-frequency inference.

THEOREM 4. *Let $n \in \mathbb{N}$, and let $\mathfrak{M} \subseteq \mathbb{TF}$. Furthermore, let $U \in \mathfrak{M}\text{-}REX_{prob}(p)$ with $1 \geq p > 1/(n+1)$. Then*

(1) *there is an* IIM *M reliably* EX*-identifying U on $\mathfrak{M}$ with frequency $1/n$;*

(2) *for any $f \in U$ the sequence $((i_k, b_k))_{k \in \mathbb{N}}$ of M's hypotheses on f is such that there are some $r \in \{0, ..., n-1\}$ and $j \in \mathbb{N}$ satisfying $i_k = j$ and $\varphi_j = f$ for almost all k with $k \equiv r$ mod n.*

Proof. The proof mainly follows Pitt's (1989) demonstration of the theorem, which actually states that $EX_{prob}(p) \subseteq EX_{team}(n)$, except for the same minor changes we made in proving Theorem 1. That means the desired machine M again simulates the EX-team $M_0, ..., M_{n-1}$. Each M_i first tests whether any path in $\mathfrak{T}_k$ exceeds a certain threshold d of error messages. In case it does, M_i outputs $(k, 0)$, increments d, and continues on f^{k+1}. Otherwise it works exactly as Pitt's machine M_i does. Finally, M outputs the guesses of the team members in a round-robin fashion. We omit the details Q.E.D.

THEOREM 5. *Let* $\mathfrak{M} \subseteq \mathbb{TF}$, $n \in \mathbb{N}$, *and let* $U \in \mathfrak{M}\text{-REX}_{\text{freq}}(p)$ *with* $1 \geqslant p > 1(n+1)$. *Then there is an* IIM M *reliably* EX-*inferring* U *with frequency* $1/n$. *Moreover, the output sequence* $((i_k, b_k))_{k \in \mathbb{N}}$ *of* M*'s hypotheses on any* $f \in U$ *behaves such that there are an* $r \in \{0, ..., n-1\}$ *and a* $j \in \mathbb{N}$ *satisfying* $i_k = j$ *and* $\varphi_j = f$ *for almost all* $k \equiv r \bmod n$.

Proof. Using Pitt's (1989) proof of Theorem stating that $\text{EX}_{\text{freq}}(p) \subseteq \text{EX}_{\text{team}}(n)$ our theorem is shown in adding the following:

Let $f \in \mathfrak{M}$, and let $k \in \mathbb{N}$. Assume $U \in \mathfrak{M}\text{-REX}_{\text{freq}}(p)(M')$, for some IIM M'. If $M'(f^k) = (i_k, 0)$ *then each team member* $M_0, ..., M_{n-1}$ again straightforwardly outputs $(i_k, 0)$. Otherwise, the team works as Pitt's team does, but accompanies each guess i_k with $b_k = 1$. The desired IIM M now simulates the team and outputs its hypotheses in rotating order. Q.E.D.

COROLLARY 6. *Let* $\mathfrak{M} \subseteq \mathbb{TF}$. *Then* $\mathfrak{M}\text{-REX}_{\text{prob}}(1/n) = \mathfrak{M}\text{-REX}_{\text{freq}}(1/n)$ *for any number* $n \geqslant 1$.

One-sided error probabilistic inference is thus completely characterized. Consequently, in the sequel it suffices to deal with frequency identification.

3.2. *Closure Properties*

In this section we show that the reliability notions introduced above preserve the closure properties originally pointed out by Minicozzi (1976) for reliable EX-identification. For all $a \in \mathbb{N} \cup \{*\}$ the $\mathfrak{M}\text{-REX}^a$ case has already been handled in Kinber and Zeugmann (1985). Thus it remains to deal with $\mathfrak{M}\text{-RBC}_{\text{freq}}(1/n)$, $\mathfrak{M}\text{-RBC}^a$, and $\mathfrak{M}\text{-REX}_{\text{freq}}(1/n)$.

The following proposition states that any reliable inference is closed under recursively enumerable unions.

PROPOSITION 1. *Let* $\mathfrak{M} \subseteq \mathbb{TF}$, *and let* $\text{ID} \in \{\text{RBC}_{\text{freq}}(1/n), \text{RBC}^a, \text{REX}_{\text{freq}}(1/n)/n \geqslant 1, n \in \mathbb{N}, a \in \mathbb{N} \cup \{*\}\}$. *Furthermore, let* $(M_i)_{i \in \mathbb{N}}$ *be a recursive enumeration of* IIMs *working in the sense of* $\mathfrak{M}$-ID. *Then there exists an* IIM M *such that* $\mathfrak{M}\text{-ID}(M) = \bigcup_{i \in \mathbb{N}} \mathfrak{M}\text{-ID}(M_i)$.

Proof. In essence we transform Minicozzi's (1976) basic idea into our setting. Let $(M_i)_{i \in \mathbb{N}}$ be a recursive enumeration of IIMs working in the sense of $\mathfrak{M}$-ID, and let $f \in \mathfrak{M}$. If a machine M_j identifies the function f then $M_j(f^k) = (i_k, 1)$ for almost all k. On the other hand, each machine M_j not inferring the function f produces infinitely many error messages, i.e., $M_j(f^k) = (i_k, 0)$ infinitely often. The wanted IIM M identifying $\bigcup_{i \in \mathbb{N}} \mathfrak{M}\text{-ID}(M_i)$ searches for an enumerated machine that eventually identifies the function f as follows:

The machine M dovetails the computation of more and more outputs of the enumerated machines. Moreover, the IIM M counts for each machine

M_i already included into the computation the number of successively produced hypotheses that uniformly contain a 1 in its second component. This number is called weight. As long as a machine M_i trusts in its current guesses the weight successively increments. If M_i produces an error message then the weight of the IIM M_i reduces to zero. After having read the initial segment f^k of the function f the machine M favors from the first k enumerated machines that one which actually has the greatest weight. In case the maximal weight is taken by at least two IIMs M_i and M_j the machine M chooses that one which has the smallest index.

We formally define the IIM M as follows:

Let $f \in \mathfrak{M}$, and $k \in \mathbb{N}$.

$M(f^k) :=$ "Compute in parallel

$$M_0(f^0), M_0(f^1), ..., M_0(f^k),$$
$$M_1(f^0), ..., M_1(f^{k-1}),$$
$$—$$
$$—$$
$$—$$
$$M_{k-1}(f^0), M_{k-1}(f^1),$$
$$M_k(f^0)$$

and assign to each IIM M_i, $i \leqslant k$, its weight, i.e., the greatest $m \leqslant k-i$ satisfying the condition that every guess of $M_i(f^{k-i-m})$, $M_i(f^{k-i-m+1})$, ..., $M_i(f^{k-i})$ uniformly contains a 1 in its second component. Choose $w(k)$ to be the smallest $i \leqslant k$ such that the IIM M_i has the greatest weight.

In case all considered machines have weight zero, output $(k, 0)$. If not:

If $w(k) = w(k-1)$, then output $M_{w(k)}(f^{k-w(k)})$. Otherwise output $(k, 0)$."

Note that M outputs an error message at least in case it favors a new machine possibly inferring f.

Now let $f \in \mathfrak{M}$ and assume $f \in \mathfrak{M}\text{-ID}(M_j)$ for some j. Hence there is an j_0 such that $M_j(f^k) = (j_k, 1)$ for any $k \geqslant j_0$. Consequently, M_j's weight grows continously for $k \geqslant j_0$. Moreover, any machine M_i not identifying f outputs infinitely many hypotheses of the form $(i_n, 0)$. Therefore, after computing an error message M_i has weight zero. Thus, for almost all k the machine M must favor exactly one of the M_j's inferring f, i.e., $M(f^k) = M_j(f^{k-j})$. Hence M identifies f, since the finite delay does not affect the frequency. On the other hand, if no machine recognizes $f \in \mathfrak{M}$ then either all machines under consideration have weight zero, or $w(k) \neq w(k-1)$ infinitely often.

The latter follows from the fact that every M_i produces an infinite number of error messages. In both cases, M also outputs an error message. Hence M works ID-reliably on $\mathfrak{M}$. Q.E.D.

The next proposition states that reliable identification is closed under finite variance. For any class $U \in \mathbb{R}$, let $[U] = \{g/g \in \mathbb{R},\ \exists f[f \in U,\ f =_* g]\}$.

PROPOSITION 2. *Let* $\mathfrak{M} \subseteq \mathbb{TF}$, *and let* $\mathrm{ID} \in \{\mathrm{RBC}_{\mathrm{freq}}(1/n),\ \mathrm{RBC}^a,\ \mathrm{REX}_{\mathrm{freq}}(1/n)/n \geq 1,\ n \in \mathbb{N},\ a \in \mathbb{N} \cup \{*\}\}$. *Then* $U \in \mathfrak{M}$-ID *implies* $[U] \in \mathfrak{M}$-ID.

Proof. The proof is analogously done as in Minicozzi (1976). Therefore we omit the proof here. Q.E.D.

3.3. *Hierarchy Results*

The main goal of this section consists in clarifying the basic identification power of reliable frequency inference on sets $\mathfrak{M}$, We confine ourselves to consider exclusively the cases $\mathfrak{M} = \mathbb{TF}$ and $\mathfrak{M} = \mathbb{R}$, since they are of basic interest. We start with several fundamental observations that give a first answer to the question of what actually can and cannot be reliably inferred with a certain frequency. Above all, in accordance with our theorems in Section 3.1, it generally suffices to deal with the discrete frequencies 1, 1/2, 1/3, ... Note that by definition $\mathbb{TF}\text{-}\mathrm{REX}_{\mathrm{freq}}(1/n) \subseteq \mathbb{TF}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n) \subseteq \mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$, for any number $n \geq 1$. In order to obtain results as sharp as possible we proceed in the sequel almost always as follows: Studying the power of reliable frequency inference we deal, whenever appropriate, with $\mathbb{TF}\text{-}\mathrm{REX}_{\mathrm{freq}}(1/n)$ or $\mathbb{TF}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$; whereas its limitations are shown in dealing with $\mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$.

First of all, we point out that reliable frequency identification is generally less powerful than the ordinary frequency inference.

THEOREM 7. $\mathrm{EX} \setminus \mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n) \neq \emptyset$ *for every number* $n \geq 1$.

Proof. Podnieks' (1974) BC-frequency hierarchy theorem directly implies that $\mathbb{R} \notin \mathrm{BC}_{\mathrm{freq}}(1/n)$, and hence $\mathbb{R} \notin \mathrm{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$, for any $n \geq 1$.

We set $U = \{f/f \in \mathbb{R},\ \varphi_{f(0)} = f\}$. Assume that $U \in \mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$. Then Proposition 2 implies that $[U] \in \mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n)$, which is a contradiction because $[U] = \mathbb{R}$. On the other hand, $U \in \mathrm{EX}$. Q.E.D.

As an immediate consequence one gets:

COROLLARY 8. *For any number* $n \geq 1$:

(1) $\mathbb{R}\text{-}\mathrm{REX}_{\mathrm{freq}}(1/n) \subset \mathrm{EX}_{\mathrm{freq}}(1/n)$,

(2) $\mathbb{R}\text{-}\mathrm{RBC}_{\mathrm{freq}}(1/n) \subset \mathrm{BC}_{\mathrm{freq}}(1/n)$.

Next we ask whether $\mathfrak{M}$-$\mathrm{REX}_{\mathrm{freq}}(1/n)$ is always properly contained in $\mathfrak{M}$-$\mathrm{RBC}_{\mathrm{freq}}(1/n)$. The affirmative answer is given by the following theorems which, beyond that, lead to a much deeper insight into the capabilities of reliable frequency inference.

THEOREM 9. $\mathbb{TF}$-RBC$\setminus$EX* $\neq \emptyset$

Proof. In the sequel we mainly use Gold's (1965) diagonal arguments. Showing that EX $\subset$ BC Barzdin (1974) has refined Gold's (1965) proof technique in such a way that he gets a BC-inferrible function class. Barzdin's (1974) technique is powerful enough to prove the theorem.

We interprete every partial recursive function as an IIM. As was already shown by Gold (1965), there is an effective procedure $g \in \mathbb{R}$ such that $EX^*(\varphi_i) \subseteq EX^*(\varphi_{g(i)})$ and $\varphi_{g(i)} \in \mathbb{R}$, for every $i \in \mathbb{N}$. For convenience we set $M_i := \varphi_{g(i)}$ for every $i \in \mathbb{N}$. Let i be given. Now we define a class U_i such that $U_i \notin EX^*(M_i)$. We set $f_i(0) = i$. Compute $n_{i_1} = M_i(\langle i \rangle)$.

For $k = 1, 2, 3, \ldots$, check by dovetailing whether (α) or (β) happens:

(α) $M_i(\langle i1^k \rangle) \neq n_{i_1}$,

(β) $M_i(\langle i0^k \rangle) \neq n_{i_1}$.

If neither (α) nor (β) happens we set $U_i = \{i0^\infty, i1^\infty\}$. Consequently, $U_i \notin EX^*(M_i)$. Now suppose a k_1 is found satisfying (α) or (β).

If (α) happens, then define $f_i(x) = 1$, for $1 \leqslant x \leqslant k_1$.

If (β) happens, then define $f_i(x) = 0$, for $1 \leqslant x \leqslant k_1$.

In both cases continue as follows:

Compute $n_{i_2} = M_i(f_i^{k_1})$. For $k = 1, 2, 3, \ldots$, check in dovetailing whether (α) or (β) happens:

(α) $M_i(\langle f_i(0) \cdots f_i(k_1)\, 1^k \rangle) \neq n_{i_2}$,

(β) $M_i(\langle f_i(0) \cdots f_i(k_1)\, 0^k \rangle) \neq n_{i_2}$.

In case neither (α) nor (β) happens we set $U_i = \{f_i(0) \cdots f_i(k_1)\, 0^\infty, f_i(0) \cdots f_i(k_1)\, 1^\infty\}$. Then we again get $U_i \notin EX^*(M_i)$. If a k_2 is found which fulfills (α) or (β) proceed as follows:

Suppose (α) happens. Define $f_i(x) = 1$, for $k_1 + 1 \leqslant x \leqslant k_1 + k_2 + 1$.

In case (β) happens define $f_i(x) = 0$, for $k_1 + 1 \leqslant x \leqslant k_1 + k_2 + 1$.

Iterate the construction. Now assume infinitely many k_j are found satisfying the relevant conditions (α) or (β). Then $f_i \in \mathbb{R}$ and the sequence $(M_i(f_i^n))_{n \in \mathbb{N}}$ does not converge . Hence in this case we set $U_i = \{f_i\}$ yielding $U_i \notin EX^*(M_i)$. Finally, let $U = \bigcup_{i \in \mathbb{N}} U_i$. Thus $U \notin EX^*$.

It remains to prove $U \in \mathbb{TF}$-RBC(M), for some IIM M. Let $f \in \mathbb{TF}$ and let $n \in \mathbb{N}$. We define:

$M(f^n) =$ "Simulate the construction of $U_{f(0)}$ in computing just the first

FIGURE 1

n levels of the tree which actually represents the possible structure of $U_{f(0)}$. This finite tree has at most two branches since one branch will be deleted whenever the relevant (α) or (β) happens. If the initial segment $f(0) \cdots f(n)$ turns out to be different from all branches just existing then output $(n, 0)$.

Otherwise, output $(e, 1)$, where e is a canonical program of the following function η. For every $x \leqslant n$ we set $\eta(x) = f(x)$. The algorithm computing η for $x > n$ works as follows:

Continue to completely simulate the construction of $U_{f(0)}$. Two cases are possible.

Case 1. No relevant (α) or (β) happens. Then the values of η are equal to thoses represented by this branch (cf. Fig. 1).

Case 2. The relevant (α) or (β) happens. Remember that one branch corresponds to the computation performed in (α) while the other one corresponds to the computation performed in (β). Now we have to distinguish between the following subcases:

Subcase 2.1. The branch coinciding with $\eta(x)$, for all $x \leqslant n$, represents the computation performed in (α), and (β) happens. In accordance with our construction the branch representing the computation performed in (α) is then deleted. As long as the computation in (α) is performed $\eta(x)$ is defined. After the deletion of that branch $\eta(x)$ is undefined (cf. Fig. 2).

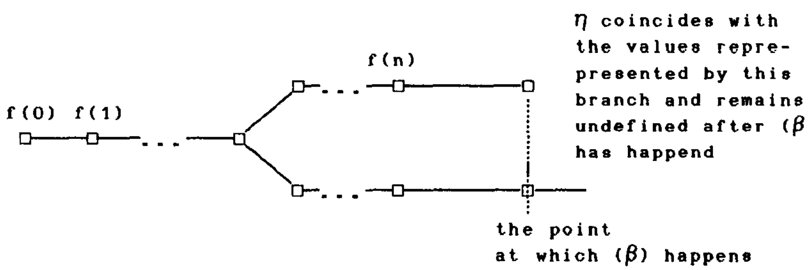

FIGURE 2

Subcase 2.2. The branch coinciding with $\eta(x)$, for all $x \leqslant n$, represents the computation performed in (β), and (α) happens. This case is handled analogously to Subcase 2.1.

Subcase 2.3. The branch coinciding with $\eta(x)$, for all $x \leqslant n$, represents the computation performed in (α), and (α) happens. Then the branch representing the computation performed in (β) is deleted. The construction is iterated; i.e., (α) and (β) are restarted with new values. The function η will not be computed further until (α) or (β) happens again. Suppose (α) or (β) happens. Then one branch will be deleted. Now the function η can be defined for all those arguments that correspond to the branch not beeing deleted, i.e., η coincides with the values between these ramification points. By iterating this construction, η can eventually be computed for more and more arguments (cf. Fig. 3).

Subcase 2.4. The branch coinciding with $\eta(x)$, for all $x \leqslant n$, represents the computation performed in (β), and (β) happens. This case can be handled analogously to Subcase 2.3."

It remains to show that the IIM M identifies U.

Claim. $U \in \mathbb{TF}\text{-RBC}(M)$

Case 1. $f \notin U$. By construction we get $f \notin U_{f(0)}$. Therefore, almost all initial segments of f differ from the actually created tree. Hence, $M(f^n) = (n, 0)$ for almost all n.

Case 2. $f \in U$. Thus we conclude $f \in U_{f(0)}$. Suppose $\text{card}(U_{f(0)}) = 2$. Then $U_{f(0)}$ is represented by a tree consisting of an initial segment which ramifies at a certain point n_0 into two infinite branches. The construction of M now ensures that M's output is correct for any $n > n_0$.

If $\text{card}(U_{f(0)}) = 1$ then M outputs only correct hypotheses. Q.E.D.

THEOREM 10. *Let* $\mathfrak{M} \in \{\mathbb{TF}, \mathbb{R}\}$, *and let* $a \in \mathbb{N} \cup \{*\}$. *Then* $\mathfrak{M}\text{-REX}^a \subset \mathfrak{M}\text{-RBC}$.

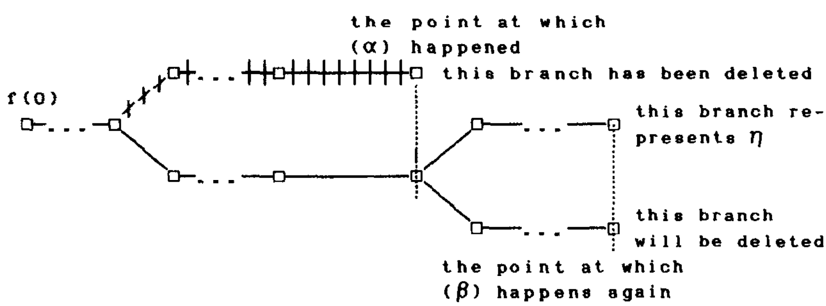

FIGURE 3

Proof. Due to the preceding theorem $\mathfrak{M}$-RBC$\backslash\mathfrak{M}$-REX$^* \neq \emptyset$ is obvious. Hence it suffices to show that $\mathfrak{M}$-REX$^a \subseteq \mathfrak{M}$-RBC, for every $a \in \mathbb{N} \cup \{*\}$. This is done as in Case and Smith (1983), whereas every mind change of M witnessing that $U \in \mathfrak{M}$-REX^a is reflected by an error message. Q.E.D.

The following corollary summerizes the results concerning reliable BC-inference.

COROLLARY 11. EX* # $\mathbb{TF}$-RBC *and* EX* # $\mathbb{R}$-RBC.

Proof. Since EX $\subset$ EX* and $\mathbb{R}$-RBC $= \mathbb{R}$-RBC$_{\text{freq}}(1)$ the corollary is an immediate consequence of Theorem 7 as well as of Theorem 9. Q.E.D.

By the next theorem we heighten the known results dealing with the capabilities of reliable EX^a-identification.

THEOREM 12. $\mathbb{TF}$-REX$^a\backslash$EX$_{\text{team}}(a) \neq \emptyset$ *for any* $a \in$ N.

Proof. The $a=1$ case has already been proved by us previously (cf. Kinber and Zeugmann, 1985). In the sequel we demonstrate how the general case can be handled. However, we present the proof for the case $a=2$ only since thereafter the generalization is straightforward. Every number i is interpreted to be just the encoding of two IIMs M_n, M_m which we want to fool. Without loss of generality we can assume that $M_n, M_m \in \mathbb{R}$ (cf. the proof of Theorem 9). Now, for every $i \in \mathbb{N}$ we effectively construct a function $f_i \in \mathbb{R}_2$ such that there is a total extension of f_i (possibly f_i itself) on which the related team M_n, M_m fails.

Let i be given. Compute n and m. We set $f_i(0)=i$. Then define $f_i(2)$, $f_i(3)$, ..., to be zero until a k_1 is found such that

(α) $M_n(\langle \mathrm{i} \rangle) \neq M_n(\langle i10^{k_1} \rangle)$, or

(β) $M_n(\langle \mathrm{i} \rangle) \neq M_n(\langle i00^{k_1} \rangle)$, or

(γ) $M_m(\langle i \rangle) \neq M_m(\langle i10^{k_1} \rangle)$, or

(δ) $M_m(\langle i \rangle) \neq M_m(\langle i00^{k_1} \rangle)$ happens.

If no such k_1 is found then f_i is already defined for all but one argument x (namely $x=1$).

If (α) or (β) happens then set $f_i(1)=1$ or $f_i(1)=0$, respectively. Furthermore, let $1=m$, and $r=n$.

If (γ) or (δ) happens then set $f_i(1)=1$ or $f_i(1)=0$, respectively, but let $1=n$, and $r=m$.

In the parameter r we remember which machine has made the mind change. In order to fool the whole team now we are mainly interested in a mind change of the machine M_1. However, M_1 may not behave thus. Therefore we introduce a second trap.

LOOP.

Set $\tau = f_i^{k_1+1}$. Now define $f_i(k_1+4)$, $f_i(k_1+5)$, ..., to be zero until a k_2 is found such that either

(α) $\varphi_{M_1(\tau)}(k_1+2)$ turns out to be defined within k_2 steps, or

β_0) $M_1(\tau) \neq M_1(\tau 0^{k_2+2})$, or

(β_1) $M_1(\tau) \neq M_1(\tau 1 0^{k_2+1})$, or

(γ_{00}) $M_r(\tau) \neq M_r(\tau 0^{k_2+2})$, or

(γ_{01}) $M_r(\tau) \neq M_r(\tau 010^{k_2})$, or

(γ_{10}) $M_r(\tau) \neq M_r(\tau 10^{k_2+1})$, or

(γ_{11}) $M_r(\tau) \neq M_r(\tau 110^{k_2})$ happens.

If nothing happens then f_i is defined for all but *two* arguments x (namely $x = k_1+2$, k_1+3).

Suppose (α) happens first. Then we set $f_i(k_1+2) = 1 \dot{-} \varphi_{M_1}(k_1+2)$ and $f_i(k_1+3) = 0$. Swap 1 and r. Goto **LOOP** and iterate the construction.

In case (β_0) or (β_1) happens first, we set $f_i(k_1+2) = 0$ or $f_i(k_1+2) = 1$, respectively, and define $f_i(k_1+3) = 0$. Swap 1 and r. Goto **LOOP** and iterate the construction.

Now we assume one of the (γ_{cd}) cases happens, where $c, d \in \{0, 1\}$. Stop all procedures. Define $f_i(k_1+3) = d$ and set $z = c$.

Remark. $f_i(k_1+2)$ remains undefined yet.

LOOP 1

Proceed as follows:

Define $f_i(k_1+k_2+6)$, $f_i(k_1+k_2+7)$, ..., to be zero until a k_3 is found such that either

(A) $\varphi_{M_1(\tau)}(k_1+2)$ turns out to be defined within k_2+k_3 steps, or

(B_0) $M_r(\tau z f_i(k_1+3)\, 0^{k_2}) \neq M_r(\tau z f_i(k_1+3)\, 0^{k_2} 0^{k_3+1})$, or

(B_1) $M_r(\tau z f_i(k_1+3)\, 0^{k_2}) \neq M_r(\tau z f_i(k_1+3)\, 0^{k_2} 1 0^{k_3})$, or

(C_0) $M_1(\tau) \neq M_1(\tau z f_i(k_1+3)\, 0^{k_2} 0^{k_3+1})$, or

(C_1) $M_1(\tau) \neq M_1(\tau 1 - z f_i(k_1+3)\, 0^{k_2} 0^{k_3+1})$ happens.

Comment. By (B_0) and (B_1) we actually search for the next mind change of machine M_r. In (C_0), (C_1) we essentially proceed to test whether the former (β_0) or (β_1) happens, but we take into consideration that meanwhile $f_i(k_1+3)$ has been defined. The earlier step (α) is replaced by (A), i.e., augmenting the number of allowed steps.

Now, if (A) happens first then we set $f_i(k_1+2)=1 \dot{-} \varphi_{M_1(\tau)}(k_1+2)$ and $f_i(k_1+k_2+5)=0$, swap r and 1, and go to **LOOP** in order to iterate the construction.

Similarly, if (C_0) or (C_1) happens first then define $f_i(k_1+2)=z$ or $f_i(k_1+2)=1-z$, respectively, and set $f_i(k_1+k_2+5)=0$. Perform the swap of r and 1, go to **LOOP**, and iterate the construction.

In case that (B_0) or (B_1) first turns out to be fulfilled we define only $f_i(k_1+k_2+5)=0$ or $f_i(k_1+k_2+5)=1$, respectively. Then we return to **LOOP** 1 and iterate the subconstruction.

Let us now discuss what different variants of f_i may occur. Suppose both machines do perform only finitely many mind changes, and for the relevant k the value $\varphi_{M_l(f_i^k)}(k+1)$ is not defined. Therefore, f_i is defined everywhere, but for two arguments. Moreover, M_1's guesses are almost everywhere equal to $M_1(f_i^k)$, which cannot be correct, since $\varphi_{M_l(f_i^k)}(k+1)$ is not defined. On the other hand, in (B_0) and (B_1) machine M_r is forced to make a mind change. Because this mind change does not occur, there is a total extension of f_i on which M_1 and M_r fail.

Next assume that only M_r performs infinitely many mind changes and again $\varphi_{M_l(f_i^k)}(k+1)$ diverges. Hence f_i is defined for all but the argument k. Let $f(x)=f_i(x)$, for all $x \neq k$, and set $f(k)=z$, where z is due to the above construction. Then the team M_n, M_m again fails.

Finally, infinitely many swaps of the appropriate 1 and r are performed. Consequently, $f_i \in \mathbb{R}$ but neither M_n not M_m succeeds.

Now we set $U=\{f/f \in \mathbb{R}$ and f is a total extension of some $f_i\}$. Obviously, $U \notin \mathrm{EX}_{\mathrm{team}}(2)$.

The wanted machine M is defined as follows: Let $t \in \mathbb{TF}$ and $x \in \mathbb{N}$.

$M(t^x)=$"Compute $f_{t(0)}(z)$ for all but at most two arguments $z \leqslant x$, where $f_{t(0)}$ is defined as in the above construction. Then test whether $t(z)=f_{t(0)}(z)$ for all but at most two arguments $z \leqslant x$. For if not, output $(x, 0)$. Otherwise, output $(e, 1)$, where e is a canonical program performing the above construction."

The verification of $U \in \mathbb{TF}\text{-REX}^2(M)$ is left to the reader. Q.E.D.

Through exploration of the above results the announced relation between the two reliable frequency identification modes (i.e., EX and BC) can now be obtained.

THEOREM 13. *Let $\mathfrak{M} \in \{\mathbb{TF}, \mathbb{R}\}$. Then*

$$\mathfrak{M}\text{-REX}_{\mathrm{freq}}(1/n) \subset \mathfrak{M}\text{-RBC}_{\mathrm{freq}}(1/n).$$

Proof. The simple inclusion $\subseteq$ is obvious. We show the proper containment by first applying Theorem 12. Hence there is a class $U \in \mathfrak{M}\text{-REX}^n \setminus \mathrm{EX}_{\mathrm{team}}(n)$. Therefore, $U \in \mathfrak{M}\text{-REX}^n \setminus \mathfrak{M}\text{-REX}_{\mathrm{freq}}(1/n)$ since

$\mathrm{EX}_{\mathrm{freq}}(1/n) = \mathrm{EX}_{\mathrm{team}}(1/n)$ (cf. Pitt, 1989) and $\mathfrak{M}\text{-REX}_{\mathrm{freq}}(1/n) \subset \mathrm{EX}_{\mathrm{freq}}(1/n)$ by Corollary 8. Due to Theorem 10 we have $U \in \mathfrak{M}\text{-RBC}$. Finally, $\mathfrak{M}\text{-RBC} \subseteq \mathfrak{M}\text{-RBC}_{\mathrm{freq}}(1/n)$ for any number $n \geqslant 1$. Q.E.D.

Smith (1982) completely relates the EX and BC team hierarchies, thereby in particular showing that $\mathrm{EX}_{\mathrm{team}}(n+1) \setminus \mathrm{BC}_{\mathrm{team}}(n) \neq \emptyset$. Unfortunately, none of his proofs can be applied in our setting. This is caused by the fact that the capabilities of pluralism are mainly based on non-union problems. Therefore, at first glance it may seem to be hopeless to transfer the power of team inference, at least partially, to reliable frequency identification. However, this is a misleading impression. Reliable frequency inference is closed under union since all the inferrible classes U share the *common property* that functions not contained in U lead to infinite sequences of error messages. Nevertheless, even in the limit it may be undecidable into which subclass of U the considered function falls. Yet we were surprised to find the next theorem.

THEOREM 14. *$\mathbb{TF}\text{-REX}_{\mathrm{freq}}(1/(n+1)) \setminus \mathrm{BC}_{\mathrm{team}}(n) \neq \emptyset$ for any number $n \geqslant 1$.*

Proof. Since the following demonstration is technically involved we explain the basic ideas in handling the case $n = 1$. Then we describe in detail how the construction can be generalized, thereby dealing with $n = 2$. The rest is straightforward.

We again interprete every partial recursive function as an IIM. In Barzdin (1974) it has been pointed out that there exists an effective procedure $h \in \mathbb{R}$ such that $\mathrm{BC}(\varphi_i) \subseteq \mathrm{BC}(\varphi_{h(i)})$ for all $i \in \mathbb{N}$, and $\varphi_{h(i)} \in \mathbb{R}$. We set $M_i = \varphi_{h(i)}$. Next we *noneffectively* construct classes U_i fulfilling $U_i \notin \mathrm{BC}(M_i)$. Hence we set $U = \bigcup_{i \in \mathbb{N}} U_i$ and obtain that $U \notin \mathrm{BC}$. As we shall see later U will be reliably EX-inferrible on $\mathbb{TF}$ with frequency $\frac{1}{2}$. Let $i \in \mathbb{N}$ be arbitrarily fixed. U_i is uniformly defined as follows:

Fix a strictly monotone function $r \in \mathbb{R}$ such that $\Phi_j = \varphi_{r(j)}$ for all $j \in \mathbb{N}$ (cf. Blum, 1967). Hence Val r is recursive. Let $g \in \mathbb{R}$ be chosen such that for all j (cf. e.g., Machtey and Young, 1978) it holds

$$\varphi_{g(j)}(0) = i$$

$$\varphi_{g(j)}(1) = \begin{cases} k, & \text{if there is a } k \text{ with } r(k) = j \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi_{g(j)}(2) = 0.$$

The definition of $\varphi_{g(j)}$ proceeds in stages. At the beginning of stage s suppose $\varphi_{g(j)}(z)$ is already defined for all $z \leqslant \sigma_{s-1}$.

We set $\sigma_0 = 2$, and for $s \geqslant 1$ we define:

Stage s. Let $\tau = \varphi_{g(j)}^{s-1}$. For $\mu = 1, 2, 3, \ldots$, dovetail the computation of $m_\mu = M_i(\tau 0^\mu)$ and $\varphi_{m_\mu}(\sigma_{s-1} + \mu + 1)$ until a $\varphi_{m_\mu}(\sigma_{s-1} + \mu + 1)$ turns out to be defined. If no such μ is found, stage s never terminates.

Now suppose a μ has been found such that $\varphi_{m_\mu}(\sigma_{s-1} + \mu + 1)_\downarrow$. We define

$$\varphi_{g(j)}(x) = \begin{cases} 0, & \text{if } \sigma_{s-1} < x \leqslant \mu + \sigma_{s-1} \\ \varphi_{m_\mu}(\sigma_{s-1} + \mu + 1) + 1, & \text{if } x = \mu + \sigma_{s-1} + 1 \\ \max\{\varphi_j(z)/z & \text{if } x = \mu + \sigma_{s-1} + 2, \\ \quad \leqslant \sigma_{s-1} + \mu + 1\} + 1, & \quad \varphi_j(z)_\downarrow \text{ for all} \\ & \quad z < \sigma_{s-1} + \mu + 1 \\ \text{undefined}, & \text{if } x = \mu + \sigma_{s-1} + 2, \\ & \quad \varphi_j(z)^\uparrow \text{ for } a \\ & \quad z < \sigma_{s-1} + \mu + 1 \\ 0, & \text{if } x = \sigma_{s-1} + \mu + 3 \end{cases}$$

$\sigma_s := \sigma_{s-1} + \mu + 3$
goto stage $s + 1$

end stage s.

By the Recursion Theorem (cf., e.g., Rogers, 1967) a number b can effectively be found such that $\varphi_{g(r(b))} = \varphi_b$. In accordance with the above construction one gets $\varphi_b(0) = i$.

Comment. Later on this will deliver the information which IIM has to be simulated.

Furthermore, $\varphi_b(1) = b$, and $\varphi_b(2) = 0$.

Claim. After stage s has been left, the values $\varphi_b(z)$ for $z \leqslant \sigma_s$ are all defined.

We prove the claim inductively. In order to leave stage s a certain μ has to be found such that $\varphi_{m_\mu}(\sigma_{s-1} + \mu + 1)$ is defined. By construction we get $\varphi_b(x) = 0$ for all $x \in \{\sigma_{s-1} + 1, \ldots, \sigma_{s-1} + \mu\}$, and therefore, from the induction hypothesis, it follows that $\Phi_b(x)$ is defined for all $x \leqslant \mu + \sigma_{s-1}$. Moreover, $\varphi_b(\sigma_{s-1} + \mu + 1) = \varphi_{m_\mu}(\sigma_{s-1} + \mu + 1) + 1$, and consequently $\Phi_b(\sigma_{s-1} + \mu + 1)$ also converges. Thus the $\max\{\varphi_{r(b)}(z)/z \leqslant \sigma_{s-1} + \mu + 1\} = \max\{\Phi_b(z)/z \leqslant \sigma_{s-1} + \mu + 1\}$ does exist. So we obtain that $\varphi_b(\sigma_{s-1} + \mu + 2)$ is defined. Finally, $\varphi_b(\sigma_{s-1} + \mu + 3) = 0$ obviously converges. This proves the claim.

If stage s is left for all $s \in \mathbb{N}$ then set $U_i = \{\varphi_b\}$. By the claim it follows that $\varphi_b \in \mathbb{R}$. On the other hand, there are infinitely many guesses m_j of the machine M_i when applied to φ_b satisfying $\varphi_{m_j} \neq \varphi_b$. Hence, M_i does not infer U_i.

Otherwise, stage s is left only finitely many times. Let s' be the last s for which stage s has been successfully finished. Now let $\tau = \varphi_b^{\sigma_{s'}}$. We set $U_i = \{\tau 0^\infty\}$. In accordance with the construction one obtains straightforwardly that almost all of M_i's hypotheses on $\tau 0^\infty$ fail to be correct. Again it follows that $U_i \notin \mathrm{BC}(M_i)$.

Please note that the above b can be computed effectively by knowing M_i only. Let $U = \bigcup_{i \in \mathbb{N}} U_i$. It remains to show that there is an IIM M reliably EX-identifying the class U on the set $\mathbb{TF}$ with frequency $\frac{1}{2}$. Let $f \in \mathbb{TF}$ and $n \in \mathbb{N}$. We define:

$M(f^n) :=$ "If $n = 0$, then output $(f(0), 1)$."

Compute from $f(0)$ the appropriate fixpoint b.

If $n = 1$, then test whether $f(1) = b$. For if not, output $(1, 0)$, and cancel f. Otherwise output $(b, 1)$.

For all $n > 1$:

If f has already been cancelled in a previous stage then output $(n, 0)$. Goto (A).

Otherwise, output $(f(1), 1)$, if n is even, and $(e_n, 1)$, if n is odd, where e_n is a canonical program of the following function η_n: Let z be the greatest $x \leqslant n$ for which $f(x) \neq 0$. Then define

$$\eta_n(y) = \begin{cases} f(y), & \text{if } y \leqslant z \\ 0, & \text{if } y > z. \end{cases}$$

Before reading the next value of f (i.e., $f(n+1)$) execute the following procedure, **CANCEL**.

CANCEL

(1) Test whether there is a number y with $0 \leqslant y \leqslant n-2$ such that all the values $f(y)$, $f(y+1)$, $f(y+2)$ are not equal to zero. If such a y has been found then cancel f, and return.

(2) Test whether there is a number $0 < y \leqslant n-1$ such that $f(y) \neq 0$ but $f(y-1) = f(y+1) = 0$.
If there is such a y then cancel f, and return.

(3) Simulate the computation of $\varphi_{f(1)}(x)$ for all $x \leqslant n$ in parallel exactly n steps. For any $\varphi_{f(1)}(x)$ turning out to be defined test whether or not $\varphi_{f(1)}(x) = f(x)$. For if not, cancel f, and return.

(4) Test whether there is a $y \leqslant n-2$ such that $f(y) \neq 0 \neq f(y+1)$ and $f(y-1) = 0 = f(y+2)$. If such a y has been detected test whether $\Phi_{f(1)}(x) \leqslant f(x+1)$ for all $x \leqslant y$. In case the inequality is not fulfilled cancel f and return. Otherwise test whether $\varphi_{f(1)}(x) = f(x)$ for all $x \leqslant y$.

Comment. $\varphi_{f(1)}(x)$ converges for all $x \leqslant y$ since $\Phi_{f(1)}(x) \leqslant f(x+1)$. If there is an $x \leqslant y$ with $\varphi_{f(1)}(x) \neq f(x)$ cancel f and return.

end **CANCEL**

(A) Request the next value of f."

We finish the proof by showing that M behaves as required.

Claim. M works reliably in the sense of $\mathbb{TF}\text{-REX}_{\text{freq}}(\frac{1}{2})$.

Let $f \in \mathbb{TF}$. Suppose M outputs for almost all n hypotheses of the type $(i_n, 1)$. We have to prove that there is a particular guess i occurring with frequency $\frac{1}{2}$ such that $\varphi_i = f$. Note that after f has been cancelled once, M outputs hypotheses of the form $(i_n, 0)$ only. Therefore, all guesses must be of the type $(i_n, 1)$. Moreover, f will never be cancelled.

Case 1. $\varphi_{f(1)} \in \mathbb{R}$. Then, in exectution (4) of procedure **CANCEL** we verify that $\varphi_{f(1)} = f$. To see this assume the converse. Let x be fixed such that $\varphi_{f(1)}(x) \neq f(x)$. Hence after $\Phi_{f(1)}(x)$ steps the function f will be cancelled; this is a contradiction. Consequently, for all even n the output of M is correct.

Case 2. $\varphi_{f(1)} \notin \mathbb{R}$. By construction there is a k_0 such that $\varphi_b(x) = \varphi_{f(1)}(x)$, and $\varphi_{f(1)}(x)_\downarrow$ for all $x \leqslant k_0$. Furthermore, $\varphi_{f(1)}(x)^\uparrow$ for all $x > k_0$. Therefore, after max $\{\Phi_{f(1)}(x)/x \leqslant k_0\}$ steps M verifies that $\varphi_{f(1)}^{k_0} = f^{k_0}$ since otherwise f would be cancelled. Now it is easy to see that $f(x)$ must be equal to zero, for all $x > k_0$, since otherwise f will be cancelled again.

Hence $(e_n, 1)$ is a correct guess for $n \geqslant k_0$; i.e., the required particular guess i is e_n at least for all $n \geqslant k_0$. Consequently, for almost all odd n the output of M is correct. This proves the claim.

The theorem follows since no function from U will be cancelled by M (cf. the above construction). This completes the proof in the case $k = 1$.

Now we consider the case $k = 2$, thereby learning how the generalization has to be performed. Any $i \in \mathbb{N}$ is interpreted as the binary encoding of just two numbers, n and m. We construct, again *noneffectively*, a class $U_i \notin \text{BC}_{\text{team}}(M_n, M_m)$. Without loss of any generality we may assume that $M_n, M_m \in \mathbb{R}$. Applying the above construction based on the Recursion Theorem we define a function φ_b as follows: $\varphi_b(0) = i$, $\varphi_b(1) = b$, $\varphi_b(2) = 0$, and for $x > 2$ we proceed in stages. However, we have to deal with two IIMs. Similarly to the proof of Theorem 12 this requires a priority regime. Using the parameter k we remember which machine has the greatest priority. Furthermore, we are again forced to deal with arguments on which φ_b will eventually not be defined. These gaps possibly change from stage to stage. Hence, in any stage s they are stored in the sets C_1, and C_2. Finally, we introduce working initial segments τ which will be updated at the end of each stage. Whenever $\varphi_b(x)_\downarrow$, for $x \leqslant \sigma_s$, then $\tau(x) = \varphi_b(x)$, where σ_s describes the length of the initial segment we are dealing with at the end of stage s. For $x \in C_1 \cup C_2$ and $x \leqslant \sigma_s$, we define $\tau(x) = 0$. Instead of pairs, where the first component was preassigned to fool the machine, and the second component was used to ensure inferribility in summarizing the complexities, now we deal with 4-tuples, i.e., one pair for each machine. Moreover, we can summarize only complexities on those arguments on which φ_b has already been defined. We use even numbers to characterize complexity bounds on all previous arguments and odd numbers in case gaps do actually occur.

Now we describe the construction formally. Let $\sigma_0 = 2$ and $\tau(x) = \varphi_b(x)$ for all $x \leqslant \sigma_0$.

Stage 1. For $\mu = 1, 2, 3, ...,$ dovetail the computation of $r(n, \mu) = M_n(\tau^{\sigma_0}0^\mu)$ and $r(m, \mu) = M_m(\tau^{\sigma_0}0^\mu)$ as well as $\varphi_{r(n,\mu)}(\sigma_0 + \mu + 1)$ and $\varphi_{r(m,\mu)}(\sigma_0 + \mu + 3)$ until $\varphi_{r(n,\mu)}(\sigma_0 + \mu + 1)$ or $\varphi_{r(m,\mu)}(\sigma_0 + \mu + 3)$ turns out to be defined for some μ.

Case 1. A μ has been found such that $\varphi_{r(n,\mu)}(\sigma_0 + \mu + 1)$ is discovered to be defined. Then set $C_1 = \{\sigma_0 + \mu + 3\}$ and $C_2 = \{\sigma_0 + \mu + 4\}$ and define

$$\varphi_b(x) = \begin{cases} 0, & \text{if } \sigma_0 < x \leqslant \sigma_0 + \mu \\ \varphi_{r(n,\mu)}(\sigma_0 + \mu + 1) + 1, & \text{if } x = \sigma_0 + \mu + 1 \\ 2 \cdot \max\{\Phi_b(z)/z \leqslant \sigma_0 + \mu + 1\} + 2, & \text{if } x = \sigma_0 + \mu + 2 \\ 0, & \text{if } x = \sigma_0 + \mu + 5. \end{cases}$$

Furthermore, let $\sigma_1 = \sigma_0 + \mu + 5$ and $k = m$. Set

$$\tau(x) = \begin{cases} \varphi_b(x), & \text{if } x \in \{0, ..., \sigma_1\} \setminus (C_1 \cup C_2) \\ 0, & \text{if } x \in C_1 \cup C_2. \end{cases}$$

goto Stage 2

Case 2. A μ has been found such that $\varphi_{r(m,\mu)}(\sigma_0 + \mu + 3)$ turns out to be defined first. Then set $C_1 = \{\sigma_0 + \mu + 1\}$, and $C_2 = \{\sigma_0 + \mu + 2\}$ and define

$$\varphi_b(x) = \begin{cases} 0, & \text{if } \sigma_0 < x \leqslant \sigma_0 + \mu \\ \varphi_{r(m,\mu)}(\sigma_0 + \mu + 3) + 1, & \text{if } x = \sigma_0 + \mu + 3 \\ 2 \cdot \max\{\Phi_b(z)/z \notin C_1 \cup C_2, & \\ \quad z \leqslant \sigma_0 + \mu + 1\} + 1, & \text{if } x = \sigma_0 + \mu + 4 \\ 0, & \text{if } x = \sigma_0 + \mu + 5. \end{cases}$$

Furthermore, let $\sigma_1 = \sigma_0 + \mu + 5$ and $k = n$. Set

$$\tau(x) = \begin{cases} \varphi_b(x), & \text{if} \quad x \in \{0, ..., \sigma_1\} \setminus (C_1 \cup C_2) \\ 0, & \text{if} \quad x \in C_1 \cup C_2. \end{cases}$$

goto Stage 2

After having described the initialization we declare stage s for $s \geqslant 2$.

Stage s. For $\mu = 1, 2, 3, ...,$ dovetail the computations $r(n, \mu) = M_n(\tau^{\sigma_s - 1}0^\mu)$, and $r(m, \mu) = M_m(\tau^{\sigma_s - 1}0^\mu)$, and try to compute

(α) $\varphi_{r(n, \mu)}(\sigma_{s-1} + \mu + 1)$ and $\varphi_{r(m, \mu)}(\sigma_{s-1} + \mu + 3)$,

(β) $\varphi_{r_y}(y)$, for any $y \in C_1$, where $r_y = M_k(\tau^{y-1})$,

until one value in (α) or (β) turns out to be defined.

Case 1. For a $y \in C_1$ the value $\varphi_{r_y}(y)$ is discovered to be defined. Stop any computation and set

$$\varphi_b(x) = 0 \qquad \text{for all} \quad x \in (C_1 \cup C_2) \setminus \{y, y+1\},$$

$$\varphi_b(y) = \varphi_{r_y}(y) + 1, \qquad \text{and} \qquad \varphi_b(y+1) = 2 \max\{\Phi_b(z)/z \leqslant y\} + 2.$$

Let $C_1 = C_2 = \varnothing$ and $\tau(x) := \varphi_b(x)$ for all $x \leqslant \sigma_{s-1}$. Restart Stage s.

Case 2. For a μ the value $\varphi_{r(n, \mu)}(\sigma_{s-1} + \mu + 1)$ turns out to be defined.

Subcase 2.1. $k = n$. Define

$$\varphi_b(x) = \begin{cases} 0, & \text{if } \sigma_{s-1} < x \leqslant \sigma_{s-1} + \mu \text{ or } x \in C_1 \cup C_2 \\ \varphi_{r(n, \mu)}(\sigma_{s-1} + \mu + 1) + 1, & \text{if } x = \sigma_{s-1} + \mu + 1 \\ 2 \cdot \max\{\Phi_b(z)/z \\ \qquad \leqslant \sigma_{s-1} + \mu + 1\} + 2, & \text{if } x = \sigma_{s-1} + \mu + 2 \\ 0, & \text{if } x = \sigma_{s-1} + \mu + 5. \end{cases}$$

Furthermore, set $\sigma_s = \sigma_{s-1} + \mu + 5$ and $C_1 = \{\sigma_{s-1} + \mu + 3\}$, $C_2 = \{\sigma_{s-1} + \mu + 4\}$, and

$$\tau(x) = \begin{cases} \varphi_b(x), & \text{if} \quad x \in \{0, ..., \sigma_s\} \setminus (C_1 \cup C_2) \\ 0, & \text{if} \quad x \in C_1 \cup C_2. \end{cases}$$

Moreover, let $k = m$.

goto Stage $s + 1$

Subcase 2.2. $k \neq n$. Define

$$\varphi_b(x) = \begin{cases} 0, & \text{if} \quad \sigma_{s-1} < x \leqslant \sigma_{s-1} + \mu \\ \varphi_{r(n,\mu)}(\sigma_{s-1} + \mu + 1) + 1, & \text{if} \quad x = \sigma_{s-1} + \mu + 1 \\ 2 \cdot \max\{\Phi_b(z)/z \notin C_1 \cup C_2, & \\ \quad z \leqslant \sigma_{s-1} + \mu + 1\} + 1, & \text{if} \quad x = \sigma_{s-1} + \mu + 2 \\ 0, & \text{if} \quad x = \sigma_{s-1} + \mu + 5. \end{cases}$$

Furthermore, set $\sigma_s = \sigma_{s-1} + \mu + 5$ and $C_1 := \{\sigma_{s-1} + \mu + 3\} \cup C_1$ as well as $C_2 := \{\sigma_{s-1} + \mu + 4\} \cup C_2$. Actualize τ as

$$\tau(x) = \begin{cases} \varphi_b(x), & \text{if} \quad x \in \{0, ..., \sigma_s\} \setminus (C_1 \cup C_2) \\ 0, & \text{if} \quad x \in C_1 \cup C_2 \end{cases}$$

and set $k = m$.

goto Stage $s + 1$

Case 3. For a μ the value $\varphi_{r(m,\mu)}(\sigma_{s-1} + \mu + 3)$ turns out to be defined first.

Subcase 3.1. $k = m$. Then we set $C_1 := C_1 \cup \{\sigma_{s-1} + \mu + 1\}$ and $C_2 := C_2 \cup \{\sigma_{s-1} + \mu + 2\}$, and define

$$\varphi_b(x) = \begin{cases} 0, & \text{if} \quad \sigma_{s-1} < x \leqslant \sigma_{s-1} + \mu \\ & \quad \text{or } x \in C_1 \cup C_2 \\ \varphi_{r(m,\mu)}(\sigma_{s-1} + \mu + 3) + 1, & \text{if} \quad x = \sigma_{s-1} + \mu + 3 \\ 2 \cdot \max\{\Phi_b(z)/z & \\ \quad \leqslant \sigma_{s-1} + \mu + 3\} + 2, & \text{if} \quad x = \sigma_{s-1} + \mu + 4 \\ 0, & \text{if} \quad x = \sigma_{s-1} + \mu + 5. \end{cases}$$

Now let $C_1 = C_2 = \varnothing$ and let $\sigma_s = \sigma_{s-1} + \mu + 5$. We set $\tau(x) = \varphi_b(x)$ for all $x \leqslant \sigma_s$. Finally, let $k = n$.

goto Stage $s + 1$

Subcase 3.2. $k \neq m$. We set $C_1 := C_1 \cup \{\sigma_{s-1} + \mu + 1\}$ and $C_2 := C_2 \cup \{\sigma_{s-1} + \mu + 2\}$. The definition of φ_b is continued as follows:

$$\varphi_b(x) = \begin{cases} 0, & \text{if} \quad \sigma_{s-1} < x \leqslant \sigma_{s-1} + \mu \\ \varphi_{r(m,\mu)}(\sigma_{s-1} + \mu + 3) + 1, & \text{if} \quad x = \sigma_{s-1} + \mu + 3 \\ 2 \cdot \max\{\Phi_b(z)/z \notin C_1 \cup C_2, & \\ \quad z \leqslant \sigma_{s-1} + \mu + 3\} + 1, & \text{if} \quad x = \sigma_{s-1} + \mu + 4 \\ 0, & \text{if} \quad x = \sigma_{s-1} + \mu + 5. \end{cases}$$

We set again $\sigma_s = \sigma_{s-1} + \mu + 5$ and and actualize τ by

$$\tau(x) = \begin{cases} \varphi_b(x), & \text{if } x \in \{0, ..., \sigma_s\} \setminus \{C_1 \cup C_2) \\ 0, & \text{if } x \in C_1 \cup C_2 \end{cases}.$$

Finally, let $k = n$.

goto Stage $s+1$

Now we define the wanted class U_i. Suppose that some stage s is never left. Then we set $U_i = \{\tau^{\sigma_{s-1}} 0^\infty\}$, and by the construction it must hold that $U_i \notin \mathrm{BC}_{\mathrm{team}}(M_n, M_m)$.

Assume that stage s is left for any s, and that $\varphi_b \in \mathbb{R}$. In accordance with our construction this directly yields $\{\varphi_b\} \notin \mathrm{BC}_{\mathrm{team}}(M_n, M_m)$. Hence we set $U_i = \{\varphi_b\}$.

Finally, suppose stage s is left infinitely often but after a certain stage s_0 the value of k remains unchanged. Then we consider the function φ_a. For $x \leqslant 2$ we set $\varphi_a(x) = \varphi_b(x)$. Then φ_a is also defined in stages, but φ_a always takes the values of the appropirate part of τ, i.e., if $\varphi_b(x)_\downarrow$, then $\varphi_b(x) = \varphi_a(x)$ for all $x \geqslant \sigma_{s_0}$, and if $\varphi_b(x)^\uparrow$, then $\varphi_a(x) = 0$. Let $\hat{\tau} = \tau^{\sigma_{s_0}}$, and let $U_i = \{\hat{\tau}\varphi_a(\sigma_{s_0}+1)\, \varphi_a(\sigma_{s_0}+2) \cdots\}$.

Consequently, $U_i \notin \mathrm{BC}_{\mathrm{team}}(M_n, M_m)$.

Now we set $U = \bigcup_{i \in \mathbb{N}} U_i$ thus obtaining $U \notin \mathrm{BC}_{\mathrm{team}}(2)$.

It remains to show that $U \in \mathbb{TF}\text{-Rex}_{\mathrm{freq}}(1/3)(M)$, for some IIM M. Let $f \in \mathbb{TF}$ and $n \in \mathbb{N}$. We define: $M(f^n) :=$ "If $n = 0$, then output $(f(0), 1)$.

Compute the appropriate fixpoint b.

If $n = 1$, then test whether $f(1) = b$. In case it is not, cancel f and output $(1, 0)$. Otherwise output $(b, 1)$.

For all $n > 1$: output

$$\begin{array}{ll} (f(1), 1), & \text{if } n \equiv 0 \bmod 3 \\ (e_n, 1), & \text{if } n \equiv 1 \bmod 3 \\ (d_n, 1), & \text{if } n \equiv 2 \bmod 3 \end{array}$$

until f has been cancelled. Thereby e_n is again a canonical program of the following function η_n. Let z be the greatest $x \leqslant n$ for which $f(x) \neq 0$. Then define $\eta_n(y) = f(y)$ for $y \leqslant z$ and $\eta_n(y) = 0$, if $y > z$. Furthermore, let d_n be a program of the function δ_n defined as follows:

Let z be the greatest $x \leqslant n$ for which $f(x) \neq 0$ and $f(x)$ is even. Then set

$$\delta_n(y) = \begin{cases} f(y), & \text{if } y \leqslant z \\ \varphi_a(y), & \text{if } y > z, \end{cases}$$

where φ_a is defined as described above.

Now it is not at all hard to generalize the procedure **CANCEL**. Hence we omit the details."

Moreover, using similar arguments as we did in the case $k=1$ one straightforwardly proves that M behaves as desired. Q.E.D.

The latter theorem directly yields infinite hierarchies of reliable frequency identification starting from $\mathfrak{M}$-REX and $\mathfrak{M}$-RBC, respectively, where $\mathfrak{M} \in \{\mathbb{TF}, \mathbb{R}\}$. However, until now we knew only a little about uniform upper limitations concerning the power of reliable frequency inference. On the other hand, the BC-frequency hierarchy is properly contained in BC* since $\mathbb{R} \in \text{BC}^*$. Hence it would be interesting to know whether $\mathfrak{M}\text{-RBC}_{\text{freq}}(1/n) \subset \mathfrak{M}\text{-RBC}^*$. For $\mathfrak{M} = \mathbb{R}$ no new insight can be expected since any IIM M which BC*-identifies $\mathbb{R}$ obviously works BC*-reliably on $\mathbb{R}$, i.e., $\mathbb{R} \in \mathbb{R}\text{-RBC}^*$. Nevertheless, the case $\mathfrak{M} = \mathbb{TF}$ seems to be promising if one looks to the next theorem.

THEOREM 15. EX $\#$ $\mathbb{TF}$-RBC*.

Proof. By Corollary 11 one immediately obtains that $\mathbb{TF}\text{-RBC}^* \setminus \text{EX} \neq \emptyset$. For the other part we show first that $\mathbb{R} \notin \mathbb{TF}\text{-RBC}^*$ and then apply Proposition 1 to $U = \{f / f \in \mathbb{R},\ \varphi_{f(0)} = f\}$.

Claim. $\mathbb{R} \notin \mathbb{TF}\text{-RBC}^*$.

Let M be any IIM working BC*-reliably on $\mathbb{TF}$. Furthermore, let $(\alpha_i)_{i \in \mathbb{N}}$ be a recursive enumeration of the functions of finite support defined on initial segments $\{0, ..., n\}$ of natural numbers. We define a function f_M such that $\{f_M\} \notin \mathbb{TF}\text{-RBC}^*(M)$ as follows: Search for the least k_1 satisfying $M(\langle \alpha_{k_1} \rangle) = (i, 0)$, i.e., after α_{k_1} has been fed to M, M produces an error message. If k_1 has been found we set $f_M(x) = \alpha_{k_1}(x)$ for all $x \in \text{domain } \alpha_{k_1}$. Let $n_1 := \max \{x / x \in \text{domain } \alpha_{k_1}\}$.

Otherwise f_M will be the totally undefined function.

Now suppose such a k_1 has been found. We iterate the construction. That means, now search for the least k_2 satisfying $M(\langle \alpha_{k_1} \alpha_{k_2} \rangle) = (e, 0)$.

In case k_2 has been found we define $f_M(n_1 + x) = \alpha_{k_2}(x)$, for all $x \in \text{domain } \alpha_{k_2}$. Otherwise f_M will not be defined further. It remains to show that $f_M \in \mathbb{R}$. Assume that $t \in \mathbb{TF} \setminus \mathbb{R}$. Therefore, $\varphi_i \neq t$ for any $i \in \mathbb{N}$. Moreover, the IIM M is supposed to work BC*-reliably on $\mathbb{TF}$. Consequently, the sequence $M(t^n))_{n \in \mathbb{N}}$ must contain infinitely many error messages. Let $(M(t^{n_j}))_{j \in \mathbb{N}}$ be the subsequence defined by $M(t^{n_j}) = (e_{n_j}, 0)$, i.e., the sequence of all guesses with the second component being equal to zero. The finite sequences $\beta_j = t(n_j + 1)\, t(n_j + 2) \cdots t(n_{j+1})$ can be regarded as functions of finite support defined on $\{0, ..., n_{j+1} - n_j\}$. Since all these functions β_j are contained in the enumeration $(\alpha_i)_{i \in \mathbb{N}}$ the search procedure must terminate infinitely often. Consequently, $f_M \in \mathbb{R}$. Q.E.D.

The next theorem states that reliable frequency identification is generally less powerful than reliable BC*-inference.

THEOREM 16. $\mathbb{TF}\text{-RBC}_{\text{freq}}(1/n) \subset \mathbb{TF}\text{-RBC}^*$ *for all* $n \geqslant 1$.

Proof. Let M be an IIM witnessing $U \in \mathbb{TF}\text{-RBC}_{\text{freq}}(1/n)$. The desired machine M' reliably BC*-inferring U on $\mathbb{TF}$ works as follows: On $f \in \mathbb{TF}$ and for $k \in \mathbb{N}$ first of all $M(f^k) = (i_k, b_k)$ is computed. If $b_k = 0$ then M' outputs $(k, 0)$. Otherwise, $M'(f^k) = (e_k, 1)$, where e_k is just the program which L. Harrington's machine outputs on f^k (cf. Case and Smith, 1983). Now it can easily be seen that M' behaves correctly. Details are omitted. The proper inclusion follows by Theorem 14. Q.E.D.

Summarizing the results pointed out above, we obtain the following hierarchy:

$$\begin{array}{ccccccc}
\text{EX}_{\text{freq}}(1) & \subset & \text{EX}_{\text{freq}}(1/2) & \subset \cdots \subset & \text{EX}_{\text{freq}}(1/n) & \subset & \text{BC}^* \\
\cup & & \cup & & \cup & & \cup\,| \\
\mathfrak{M}\text{-REX}_{\text{freq}}(1) & \subset & \mathfrak{M}\text{-REX}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathfrak{M}\text{-REX}_{\text{freq}}(1/n) & \subset & \mathfrak{M}\text{-RBC}^* \\
\cap & & \cap & & \cap & & \cap\,| \\
\mathfrak{M}\text{-RBC}_{\text{freq}}(1) & \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/n) & \subset & \mathfrak{M}\text{-RBC}^* \\
\cap & & \cap & & \cap & & \cap\,| \\
\text{BC}_{\text{freq}}(1) & \subset & \text{BC}_{\text{freq}}(1/2) & \subset \cdots \subset & \text{BC}_{\text{freq}}(1/n) & \subset & \text{BC}^*
\end{array}$$

However, several questions remain open. We shall discuss them in the final section of this paper.

3.4. *Conclusions and Open Problems*

A new notion of probability inference, as well as a new concept of reliable identification, was introduced. These new types of inference were related to each other as well as to previously defined modes of identification. We characterized one-sided error probabilistic inference to coincide with reliable frequency identification. Furthermore, four new infinite hierarchies were established. We have compared them one to the other. However, in performing this comparison we did not fully succeed. It remained open whether $\mathbb{TF}\text{-REX}_{\text{freq}}(1/n) \subset \mathbb{R}\text{-REX}_{\text{freq}}(1/n)$ as well as whether $\mathbb{TF}\text{-RBC}_{\text{freq}}(1/n) \subset \mathbb{R}\text{-RBC}_{\text{freq}}(1/n)$ for any $n \geqslant 2$. For $n = 1$ the result concerning the EX case can be found in Kinber and Zeugmann (1985). Since our technique of proof does not seem to be extendable to any $n \geqslant 2$ we omit the demonstration of $\mathbb{TF}\text{-RBC} \subset \mathbb{R}\text{-RBC}$. Moreover, it would be desirable to characterize reliable frequency identification in terms of complexity theory. After the pioneering paper of Blum and Blum (1975) several identification types were shown to be complexity theoretically characterizable (cf. Wiehagen, 1978; Zeugmann, 1983). These characterizations generally led to a deeper insight as to what can actually be inferred.

The next open problem concerns reliable BC^a-inference on $\mathbb{R}$ and on $\mathbb{TF}$. Using ideas from Daley (1983) one easily shows that $\mathfrak{M}\text{-RBC}^a \subseteq \mathfrak{M}\text{-RBC}_{\text{freq}}(1/(a+1))$ for any $a \in \mathbb{N}$. Nevertheless, the most interesting question, whether or not reliable BC^a-identification is properly contained in reliable BC^{a+1}-identification, remains open. The problem would be solved if one obtained the BC-analogue of Theorem 12. We have no idea at all how to attack this difficult problem. Finally, Chen's (1981; 1982) results suggest two interesting open questions. First, Chen (1982) showed that, with the cost of finitely many anomalies, nearly minimal size programs can be inferred without reducing the power of EX^*-identification. It would be interesting to know whether this result can be extended to reliable EX^*-identification on $\mathbb{R}$ and $\mathbb{TF}$. Second, Chen (1981) proved that, for every class $U \in EX^a$, there is a class $U' \supseteq U$ such that $U' \in EX^{a+1} \setminus EX^a$. Does this theorem remain valid if EX^a is replaced by $\mathfrak{M}\text{-REX}^a$, $\mathfrak{M}\text{-RBC}^a$, $\mathfrak{M}\text{-REX}_{\text{freq}}(1/a)$, and $\mathfrak{M}\text{-RBC}_{\text{freq}}(1/a)$, where $\mathfrak{M} \in \{\mathbb{TF}, \mathbb{R}\}$?

Summary

One-Sided Error Probabilistic Inference and Reliable Frequency Identification

$$\begin{array}{ccccccc}
\mathbb{TF}\text{-REX}_{\text{freq}}(1) & \subset & \mathbb{TF}\text{-REX}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathbb{TF}\text{-REX}_{\text{freq}}(1/n) & \subset & \cdots \\
\| & & \| & & \| & & \\
\mathbb{TF}\text{-REX}_{\text{prob}}(1) & \subset & \mathbb{TF}\text{-REX}_{\text{prob}}(1/2) & \subset \cdots \subset & \mathbb{TF}\text{-REX}_{\text{prob}}(1/n) & \subset & \cdots \\
\cap & & \cap\!| & & \cap\!| & & \\
\mathbb{R}\text{-REX}_{\text{prob}}(1) & \subset & \mathbb{R}\text{-REX}_{\text{prob}}(1/2) & \subset \cdots \subset & \mathbb{R}\text{-REX}_{\text{prob}}(1/n) & \subset & \cdots \\
\| & & \| & & \| & & \\
\mathbb{R}\text{-REX}_{\text{freq}}(1) & \subset & \mathbb{R}\text{-REX}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathbb{R}\text{-REX}_{\text{freq}}(1/n) & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\mathbb{R}\text{-RBC}_{\text{freq}}(1) & \subset & \mathbb{R}\text{-RBC}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathbb{R}\text{-RBC}_{\text{freq}}(1/n) & \subset & \cdots \\
\| & & \| & & \| & & \\
\mathbb{R}\text{-RBC}_{\text{prob}}(1) & \subset & \mathbb{R}\text{-RBC}_{\text{prob}}(1/2) & \subset \cdots \subset & \mathbb{R}\text{-RBC}_{\text{prob}}(1/n) & \subset & \cdots \\
\cup & & \cup\!| & & \cup\!| & & \\
\mathbb{TF}\text{-RBC}_{\text{prob}}(1) & \subset & \mathbb{TF}\text{-RBC}_{\text{prob}}(1/2) & \subset \cdots \subset & \mathbb{TF}\text{-RBC}_{\text{prob}}(1/n) & \subset & \cdots \\
\| & & \| & & \| & & \\
\mathbb{TF}\text{-RBC}_{\text{freq}}(1) & \subset & \mathbb{TF}\text{-RBC}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathbb{TF}\text{-RBC}_{\text{freq}}(1/n) & \subset & \cdots
\end{array}$$

Reliable Inference and Its Relations to Team Identification

$$
\begin{array}{ccccccc}
\mathfrak{M}\text{-REX}^0 & \subset & \mathfrak{M}\text{-REX}^1 & \subset \cdots \subset & \mathfrak{M}\text{-REX}^a & \subset \cdots \subset & \mathfrak{M}\text{-REX}^* \\
\cap & & \cap & & \cap & & \\
\text{EX}_{\text{team}}(1) & \subset & \text{EX}_{\text{team}}(2) & \subset \cdots \subset & \text{EX}_{\text{team}}(a+1) & \subset \cdots & \\
\| & & \| & & \| & & \\
\text{EX}_{\text{freq}}(1) & \subset & \text{EX}_{\text{freq}}(1/2) & \subset \cdots \subset & \text{EX}_{\text{freq}}(1/(a+1)) & \subset \cdots & \\
\# & & \# & & \# & & \\
\mathfrak{M}\text{-REX}^1 & \subset & \mathfrak{M}\text{-REX}^2 & \subset \cdots \subset & \mathfrak{M}\text{-REX}^{a+1} & \subset \cdots \subset & \mathfrak{M}\text{-REX*} \\
\cap & & \cap & & \cap & & \cap \\
\mathfrak{M}\text{-RBC} & \subset & \mathfrak{M}\text{-RBC}^1 & \subset \cdots \subset & \mathfrak{M}\text{-RBC}^a & \subset \cdots \subset & \mathfrak{M}\text{-RBC*} \\
\cap\vert & & \cap\vert & & \cap\vert & & \\
\mathfrak{M}\text{-RBC}_{\text{freq}}(1) & \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/(a+1)) & \subset \cdots \subset & \mathfrak{M}\text{-RBC*} \\
\cup & & \cup & & \cup & & \| \\
\mathfrak{M}\text{-REX}_{\text{freq}}(1) & \subset & \mathfrak{M}\text{-REX}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathfrak{M}\text{-REX}_{\text{freq}}(1/(a+1)) & \subset \cdots \subset & \mathfrak{M}\text{-RBC*} \\
\cap & & \# & & \# & & \\
\text{BC} & = & \text{BC}_{\text{team}}(1) & \subset \cdots \subset & \text{BC}_{\text{team}}(1/a) & \subset \cdots \subset & \text{BC*} \\
\cup & & \# & & \# & & \\
\mathfrak{M}\text{-RBC}_{\text{freq}}(1) & \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/2) & \subset \cdots \subset & \mathfrak{M}\text{-RBC}_{\text{freq}}(1/(1+a)) & \subset \cdots \subset & \mathfrak{M}\text{-RBC*}
\end{array}
$$

Acknowledgment

The authors are very grateful to the reviewer for his careful reading and many valuable and helpful comments on the preparation of this paper.

RECEIVED October 21, 1988; FINAL MANUSCRIPT RECEIVED November 7, 1989

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