Modeling Incremental Learning from Positive Data

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Abstract

The present paper deals with a systematic study of incremental learning algorithms. The general scenario is as follows. Let c be any concept; then every infinite sequence of elements exhausting c is called *positive presentation* of c. An algorithmic learner successively takes as input one element of a positive presentation as well as its previously made hypothesis at a time, and outputs a new hypothesis about the target concept. The sequence of hypotheses has to converge to a hypothesis correctly describing the concept to be learned, i.e., after some point, the learner stabilizes to an accurate hypothesis. This basic scenario is referred to as *iterative learning*.

We refine this scenario by formally defining and investigating *bounded example memory inference* and *feed-back* identification. Bounded example memory and feed-back learning generalizes iterative inference by allowing to store an *a priori* bounded number of carefully chosen examples and asking whether or not a particular element did already appear in the data provided so far, respectively.

Our results are manyfold. A sufficient condition for iterative learning is provided that allows *non-enumerative* learning. We relate the learning power of our models to one another, and establish an infinite hierarchy of bounded example memory inference in dependence on the number of examples the learner is allowed to store. These results nicely contrast previously made attempts to enlarge the learning capabilities of iterative learners (cf. [16]). In particular, these results provide strong evidence that incremental learning is the art of knowing what to overlook. Moreover, feed-back learning is more powerful than iterative inference, and its learning power is incomparable to that of bounded example memory inference. Hence, there is no unique way to design superior incremental learning algorithms.

Key words: Algorithmic Learning Theory, Computational issues in A.I., Formal language learning

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1. Introduction

One of the main topics in cognitive science, epistemology, linguistic and psycholinguistic theory as well as of machine learning and algorithmic learning theory is language acquisition. The human ability to acquire their mother tongue as well as other languages has attracted a huge amount of interest in all these scientific disciplines. In particular, the main goal of the research undertaken is to gain a better understanding of what learning really is. Human language learning can be also considered as a an important example of incremental learning. However, the human ability to learn is by no means restricted to languages. Therefore, we consider in the present paper general systems that map evidence on a concept into hypotheses about it. We deal with scenarios in which the sequence of hypotheses *stabilizes* to an *accurate* and *finite* description of the target concept. Consequently, after having seen only finitely many data of the possibly infinite target, the algorithmic device performing the mapping of the data to hypotheses reaches its (generally unknown) point of convergence to a correct and finite description of the target concept. Clearly, then some form of learning must have taken place. Formalizing the notions "evidence," "stabilization," and "accuracy" results in the model of learning in the limit introduced by Gold [7]. During the last three decades much has been learned about the classes of formal languages and partial recursive functions that can successfully learned within Gold's [7] model and variations thereof (cf., e.g., [16, 21, 22, 27]). We continue along these lines of research, i.e., we study learning in the limit, too. In particular, we aim to investigate the principal learning capabilities of learners which perform incremental learning.

For the purpose of motivation and discussion of our research, we introduce some notations. A *positive presentation* of a concept c is an infinite sequence of elements that eventually exhausts all and only the elements of c. An algorithmic learner, henceforth called *inductive inference machine* (abbr. IIM), takes as input initial segments of a positive presentation, and outputs, from time to time, a hypothesis about the target concept. The set $\mathcal{H}$ of all admissible hypotheses is called *hypothesis space*. Furthermore, the sequence of hypotheses has to converge to a hypothesis correctly describing the concept to be learned, i.e., after some point, the IIM stabilizes to an accurate hypothesis. If there is an IIM that learns a concept c from all positive presentations for it, then c is said to be *learnable in the limit* with respect to the hypothesis space $\mathcal{H}$ (cf. Definition 1).

However, this model makes the somehow unrealistic assumption that the learner has access to the whole initial segment of a positive presentation provided so far. Clearly, each practical learning system has to deal with the limitations of space. Therefore, we formally define and investigate variations of the general approach described above that restrict the accessibility of input data. In particular, we deal with *iterative* learning, *bounded example memory* inference, and *feed-back* identification (cf. Definitions 3, 4, 5). All these models formalize *incremental learning*, a topic attracting more and more attention in the machine learning community (cf., e.g. [5, 19]). An iterative learner is required to produce its actual guesses exclusively from its previous one and the next element in the positive presentation. Results concerning this learning model can be found in [6, 10, 11, 16, 17, 22, 23, 27]. Osherson et al. [16] also considered the variant that the learners has access to the *last* k elements, where k is *a priori* fixed. Interestingly enough, the latter approach

does *not* increase the learning power. Alternatively, we study learners that are allowed to store k *carefully chosen* examples, where k is again *a priori* fixed (bounded example memory inference). Now, we obtain an *infinite* hierarchy of more and more powerful learners (cf. Theorem 6). This result provides strong evidence that learning is the art of *knowing what to overlook*. A similar approach has been undertaken by Ameur [1] who refined Angluin's [3] on-line learning model.

Finally, we study feed-back identification. In this setting, the iterative learner is additionally allowed to ask whether or not a particular element did already appear in the data provided so far. Again, the learning power considerably increases but the supplementary learning power is incomparable to those of bounded example memory inference. The latter result provides strong evidence that there is no unique way to design superior space efficient inference procedures.

2. Formalizing Incremental Learning

By $\mathbb{N} = \{0, 1, 2, ...\}$ we denote the set of all natural numbers. We set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Let $\varphi_0,\ \varphi_1,\ \varphi_2, ...$ denote any fixed ***programming system*** of all (and only all) partial recursive functions over $\mathbb{N}$, and let $\Phi_0,\ \Phi_1,\ \Phi_2, ...$ be any associated ***complexity measure*** (cf. Machtey and Young [15]).

By $\langle ., . \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we denote ***Cantor's pairing function***. Moreover, we use π_1 and π_2 to denote the ***projection functions*** over $\mathbb{N} \times \mathbb{N}$ to the first and second component, respectively. That is, $\pi_1\langle x, y\rangle = x$ and $\pi_2\langle x, y\rangle = y$ for all $x, y \in \mathbb{N}$.

Any discrete set $\mathcal{X}$ is called a ***learning domain***. By $\wp(\mathcal{X})$ we denote the power set of $\mathcal{X}$. Let $\mathcal{C} \subseteq \wp(\mathcal{X})$, and let $c \in \mathcal{C}$; then we refer to $\mathcal{C}$ and c as to a ***concept class*** and a ***concept***, respectively. Let c be a concept, and let $t = x_0, x_1, x_2, ...$ an infinite sequence of elements from c such that $\text{range}(t) = \{x_k \mid k \in \mathbb{N}\} = c$. Then t is said to be a ***positive presentation*** or, synonymously, a ***text*** for c. By $pos(c)$ we denote the set of all positive presentations of c. Moreover, let t be a positive presentation, and let y be a number. Then, t_y denotes the initial segment of t of length $y+1$, and $t_y^+ =_{df} \{x_k \mid k \leq y\}$. Furthermore, let $\sigma = x_0, \ldots, x_n$ be any finite sequence. Then we use $|\sigma|$ to denote the ***length*** of σ. Additionally, we use $\sigma \cdot t$ to denote the positive presentation obtained by concatenating σ and t provided $\sigma^+ \subseteq \text{range}(t)$.

In the sequel we deal with the learnability of indexable concept classes with uniformly decidable membership defined as follows (cf. Angluin [2]). A class of non-empty concepts $\mathcal{C}$ is said to be an ***indexable class*** with uniformly decidable membership provided there are an effective enumeration $c_0, c_1, c_2, ...$ of all and only the concepts in $\mathcal{C}$ and a recursive function f such that for all $j \in \mathbb{N}$ and all elements $x \in \mathcal{X}$ we have

$$f(j, x) = \begin{cases} 1, & \text{if} \quad x \in c_j, \\ 0, & \text{otherwise.} \end{cases}$$

In the following we refer to indexable classes with uniformly decidable membership as to indexable classes for short. Next, we describe some well-known examples of indexable classes. First, let Σ denote any fixed finite alphabet of symbols, and let Σ^* be the free monoid over Σ. We set $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$, where ε denotes the empty string. Then $\mathcal{X} = \Sigma^*$ serves as the learning domain. As usual, we refer to subsets $L \subseteq \Sigma^*$ as to languages (instead of concepts). Then, the set of all context sensitive languages, context free

languages, regular languages, and of all pattern languages, respectively, form indexable classes (cf. [9, 2]).

Next, let $X_n = \{0,1\}^n$ be the set of all n-bit Boolean vectors. We consider $\mathcal{X} = \bigcup_{n \geq 1} X_n$ as learning domain. Then, the set of all concepts expressible as a monomial, a k-CNF, a k-DNF, and a k-decision list form indexable classes (cf. [20, 18]).

As in Gold [7] we define an ***inductive inference machine*** (abbr. IIM) to be an algorithmic device which works as follows: The IIM takes as its input larger and larger initial segments of a positive presentation t and it either requests the next input element, or it first outputs a hypothesis, i.e., a number encoding a certain computer program, and then it requests the next input element.

The indices output by an IIM are interpreted with respect to a suitably chosen hypothesis space $\mathcal{H}$. Since we exclusively deal with indexable classes $\mathcal{C}$ we always take as a hypothesis space an indexable class $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$. Note that the indices are regarded as suitable finite encodings of the concepts described by the hypotheses. When an IIM outputs a number j, we interpret it to mean that the machine is hypothesizing h_j. Note that $\mathcal{H}$ must be defined over some learning domain $\mathcal{Z}$ comprising the learning domain $\mathcal{X}$ over which $\mathcal{C}$ is defined, and, moreover, $\mathcal{H}$ must *comprise* the target concept class $\mathcal{C}$. We say that a hypothesis $h \in \mathcal{H}$ describes a concept $c \in \mathcal{C}$ iff $c = h$, i.e., for all $z \in \mathcal{Z}$, $z \in h$ if and only if $z \in c$.

Let t be a positive presentation, and let $y \in \mathbb{N}$. Then we use $M(t_y)$ to denote the last hypothesis produced by M when successively fed t_y. The sequence $(M(t_y))_{y \in \mathbb{N}}$ is said to ***converge in the limit*** to the number j if and only if either $(M(t_y))_{y \in \mathbb{N}}$ is infinite and all but finitely many terms of it are equal to j, or $(M(t_y))_{y \in \mathbb{N}}$ is non-empty and finite, and its last term is j. Now we define some models of learning. We start with learning in the limit.

Definition 1 (Gold [7]). *Let $\mathcal{C}$ be an indexable class, let c be a concept, and let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space.* ***An IIM M LIM–identifies c from positive data with respect to $\mathcal{H}$*** *iff for every positive presentation t for c, there exists a $j \in \mathbb{N}$ such that the sequence $(M(t_y))_{y \in \mathbb{N}}$ converges in the limit to j and $c = h_j$.*

Furthermore, M LIM–identifies $\mathcal{C}$ with respect to $\mathcal{H}$ iff, for each $c \in \mathcal{C}$, M LIM–identifies c from positive data with respect to $\mathcal{H}$.

Finally, let LIM denote the collection of all indexable classes $\mathcal{C}$ for which there are an IIM M and a hypothesis space $\mathcal{H}$ such that M LIM–identifies $\mathcal{C}$ with respect to $\mathcal{H}$.

In the above definition *LIM* stands for "limit." Suppose, an IIM identifies some concept c. That means, after having seen only finitely many data of c the IIM reached its (unknown) point of convergence and it computed a *correct* and *finite* description of the target concept. Hence, some form of learning must have taken place. Therefore, we use the terms *infer* and *learn* as synonyms for identify.

Within the next definition we consider the restriction that the IIM is never allowed to output hypotheses describing proper supersets of the target concept. Inductive inference machines behaving thus are called ***conservative***.

Definition 2 (Angluin [2]). *Let $\mathcal{C}$ be an indexable class, let c be a concept, and let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space.* ***An IIM M CONSV–identifies c from positive data with respect to $\mathcal{H}$*** *iff*

(1) *M LIM–identifies c from positive data with respect to $\mathcal{H}$,*

(2) *for every positive presentation t for c and for all $y, k \in \mathbb{N}$, if $M(t_y) \neq M(t_{y+k})$ then $t^+_{y+k} \not\subseteq h_{M(t_y)}$.*

Finally, M CONSV–identifies $\mathcal{C}$ with respect to $\mathcal{H}$ iff, for each $c \in \mathcal{C}$, M CONSV–identifies c from positive data with respect to $\mathcal{H}$.

By CONSV we denote the collection of all indexable classes $\mathcal{C}$ for which there are an IIM M and a hypothesis space $\mathcal{H}$ such that M CONSV–identifies $\mathcal{C}$ with respect to $\mathcal{H}$.

Looking at the above definitions, we see that an IIM M has always access to the whole history of the learning process, i.e., in order to compute its actual guess M is fed all examples seen so far. In contrast to that, next we define ***iterative IIMs*** and a natural generalization of them called ***bounded example memory IIMs***. An iterative IIM is only allowed to use its last guess and the next element in the positive presentation of the target concept for computing its actual guess. Conceptionally, an iterative IIM M defines a sequence $(M_n)_{n \in N}$ of machines each of which takes as its input the output of its predecessor. Hence, the IIM M has always to produce a hypothesis.

Definition 3 (Wiehagen [23]). *Let $\mathcal{C}$ be an indexable class, let c be a concept, and let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space.* ***An IIM M IT–identifies c from positive data with respect to $\mathcal{H}$*** *iff for every positive presentation $t = (x_j)_{j \in \mathbb{N}}$ the following conditions are satisfied:*

(1) *for all $n \in \mathbb{N}$, $M_n(t)$ is defined, where $M_0(t) =_{df} M(x_0)$ and for all $n \geq 0$: $M_{n+1}(t) =_{df} M(M_n(t), x_{n+1})$,*

(2) *the sequence $(M_n(t))_{n \in \mathbb{N}}$ converges in the limit to a number j such that $c = h_j$.*

Finally, M IT–identifies $\mathcal{C}$ with respect to $\mathcal{H}$ iff, for each $c \in \mathcal{C}$, M IT–identifies c with respect to $\mathcal{H}$.

The resulting learning type IT is analogously defined as above.

In the latter definition $M_n(t)$ denotes the last, i.e., $(n+1)$st hypothesis output by M when successively fed the positive presentation t. Since M has to output a hypothesis in each learning step, it is justified to make the following convention. Let $\sigma = x_0, \ldots, x_n$ be any finite sequence of elements over the relevant learning domain. Moreover, let $\mathcal{C}$ be any concept class over $\mathcal{X}$, and let M be any IIM that iteratively learns $\mathcal{C}$. Then we denote by $M_y(\sigma)$ the last hypothesis output by M when successively fed σ provided $y \leq n$, and there exists a concept $c \in \mathcal{C}$ with $\sigma^+ \subseteq c$. We adopt this convention to the learning types defined below.

Within the following definition we consider a natural relaxation of iterative learning which we call bounded example memory inference. Now, an IIM M is allowed to memorize an *a priori* bounded number of the examples it already has had access to during the learning process. Again, M defines a sequence $(M_n)_{n \in \mathbb{N}}$ of machines each of which takes as input the output of its predecessor. Consequently, a bounded example memory IIM has to output a hypothesis as well as a subset of the set of examples seen so far.

Definition 4. *Let $k \in \mathbb{N} \cup \{*\}$, let $\mathcal{C}$ be an indexable class, let c be a concept, and let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space.* ***An IIM M BEM$_k$–identifies c from positive data with respect to $\mathcal{H}$*** *iff for every positive presentation $t = (x_j)_{j \in \mathbb{N}}$ the following conditions are satisfied:*

(1) *for all* $n \in \mathbb{N}$, $M_n(t)$ *is defined, where* $M_0(t) =_{df} M(x_0) = \langle j_0, S_0 \rangle$ *such that* $S_0 \subseteq t_0^+$ *and* $\mathrm{card}(S_0) \leq k$, *and for all* $n \geq 0$: $M_{n+1}(t) =_{df} M(M_n(t), x_{n+1}) = \langle j_{n+1}, S_{n+1} \rangle$ *such that* $S_{n+1} \subseteq S_n \cup \{x_{n+1}\}$ *and* $\mathrm{card}(S_{n+1}) \leq k$,
(Note that $k = *$ *means at most finitely many.)*

(2) *the sequence* $(\pi_1\langle j_n, S_n \rangle)_{n \in \mathbb{N}}$ *of* M*'s guesses converges in the limit to a number* j *such that* $c = h_j$.

Finally, M BEM_k*–identifies* $\mathcal{C}$ *with respect to* $\mathcal{H}$ *iff, for each* $c \in \mathcal{C}$, M BEM_k*–identifies* c *with respect to* $\mathcal{H}$.

For every $k \in \mathbb{N}$, the resulting learning type BEM_k is analogously defined as above. By definition, $IT = BEM_0$ as well as $BEM_* = LIM$.

Finally, we define learning by feed-back IIMs. The idea of feed-back learning goes back to Wiehagen [23] who considered it in the setting of inductive inference of recursive functions. However, his definition cannot be directly applied to learning from positive data. Informally, a feed-back IIM M is an iterative IIM that is additionally allowed to ask a particular type of questions. In each learning Stage $n+1$ M has access to the actual input x_{n+1}, and its previous guess j_n. However, M is additionally allowed to compute a query from x_{n+1} and j_n. The query concerns the history of the learning process. That is, an element x and a "YES/NO" answer A are computed such that $A = 1$ iff $x \in t_n^+$ and $A = 0$, otherwise. Intuitively, M can just ask whether or not a particular string has already been presented in previous learning stages.

Definition 5. *Let* $\mathcal{C}$ *be an indexable class, let* c *be a concept, and let* $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ *be a hypothesis space. Moreover, let* $Q{:}\,\mathcal{X} \times \mathbb{N} \to \mathcal{X}$, *and* $A{:}\,\mathcal{X} \to \{0,1\}$ *be computable total mappings.*

An IIM M ***FB–identifies*** c ***from positive data with respect to*** $\mathcal{H}$ *iff for every positive presentation* $t = (x_j)_{j \in \mathbb{N}}$ *the following conditions are satisfied:*

(1) *for all* $n \in \mathbb{N}$, $M_n(t)$ *is defined, where* $M_0(t) =_{df} M(x_0)$ *and for all* $n \geq 0$: $M_{n+1}(t) =_{df} M(M_n(t), A(Q(M_n(t), x_{n+1})), x_{n+1})$,

(2) *the sequence* $(M_n(t))_{n \in \mathbb{N}}$ *converges in the limit to a number* j *such that* $c = h_j$ *provided that* A *truthfully answers the questions computed by* Q.

Finally, M *FB–identifies* $\mathcal{C}$ *with respect to* $\mathcal{H}$ *iff there are computable mappings* Q *and* A *as described above such that, for each* $c \in \mathcal{C}$, M *FB–identifies* c *with respect to* $\mathcal{H}$.

3. Results

In this section we relate the learning power of all the models introduced to one another. In particular, we deal with the limitations of all models of incremental learning by comparing their learning power with conservative inference. Moreover, we provide results showing that rich concepts classes are incrementally learnable.

3.1. On the Limitations of Incremental Learning

All the models of incremental learning introduced above pose serious restrictions to the accessibility of data provided during the learning process. Therefore, one might readily expect a certain loss of learning power, i.e., $IT \subset LIM$, $FB \subset LIM$ as well as $BEM_k \subset LIM$ for all $k \in \mathbb{N}$. As far as iterative learning is concerned, this has been rigorously

proved in Lange and Zeugmann [11]. Hence, the more interesting question is *how much* learning power is actually lost. Answering this question, we have to take into account that learnability has been defined with respect to *suitably chosen* hypothesis spaces. As pointed out in Lange and Zeugmann [12], conservative learning is sensitive with respect to the set of allowed hypothesis spaces and so is iterative learning (cf. [27]). Therefore, it is appropriate to illustrate this dependence which is done by our next theorem.

Theorem 1. *There are an indexable class $\mathcal{C}$ and a hypothesis space $\mathcal{H}$ for it such that*

(1) *$\mathcal{C}$ is iteratively learnable with respect to $\mathcal{H}$, and*

(2) *no conservative IIM can infer $\mathcal{C}$ with respect to $\mathcal{H}$.*

Proof. We define the desired indexable class $\mathcal{C}$ via the following enumeration of languages $\mathcal{L} = (L_{\langle k,j\rangle})_{k,j\in\mathbb{N}}$. Without loss of generality, we may assume that $\Phi_k(k) \geq 1$ for all $k \in \mathbb{N}$. Now, let $k, j \in \mathbb{N}$; we distinguish the following cases.

Case 1. $\neg\, \Phi_k(k) \leq j$

Then, we set $L_{\langle k,j\rangle} = \{a^k b^\ell a^m \mid \ell, m \in \mathbb{N}^+\}$.

Case 2. $\Phi_k(k) \leq j$

Set $d = j - \Phi_k(k) + 1$.

Subcase 2.1. $d \leq \Phi_k(k)$

Let $L_{\langle k,j\rangle} = \{a^k b^\ell a^m \mid 1 \leq \ell \leq d,\ m \in \mathbb{N}^+\}$.

Subcase 2.2. $d > \Phi_k(k)$

Now, set $L_{\langle k,j\rangle} = L_{\langle k,0\rangle}$.

Since the predicate '$\Phi_i(x) \leq y$' is recursive in i, y, and z, membership is uniformly decidable with respect to the enumeration $\mathcal{L} = (L_{\langle k,j\rangle})_{k,j\in\mathbb{N}}$. We set $\mathcal{C} = \text{range}(\mathcal{L})$, and take $\mathcal{L}$ as the desired hypothesis space $\mathcal{H}$.

Claim 1. *There is an iterative IIM M learning $\mathcal{C}$ with respect to $\mathcal{L}$.*

We define an iterative IIM M which infers $\mathcal{C}$ with respect to the hypothesis space $\mathcal{L}$. Let $L \in \mathcal{C}$ and let $t = (s_j)_{j\in\mathbb{N}} \in pos(L)$. The IIM M is defined in stages, where Stage n conceptually describes M_n.

Stage 0. M receives as input s_0.

Determine the unique $k \in \mathbb{N}$ such that $s_0 = a^k b^\ell a^m$ for some $\ell,\ m \in \mathbb{N}^+$. Set $j_0 = \langle k, 0\rangle$, output j_0, and goto Stage 1.

Stage n, $n \geq 1$. M receives as input j_{n-1} and the $(n+1)$st element s_n of t.

Determine the $k, \ell, m \in \mathbb{N}$ such that $s_n = a^k b^\ell a^m$.

Case 1. $j_{n-1} = \langle k, 0\rangle$

Test whether or not $\Phi_k(k) \leq m$. In case it is, set $j_n = \langle k, \Phi_k(k)\rangle$. Otherwise, set $j_n = j_{n-1}$.

Case 2. $j_{n-1} = \langle k, z\rangle$ for some $z \in \mathbb{N}^+$

Determine $y = \Phi_k(k)$. Test whether or not $s_n = a^k b^\ell a^m \in L_{\langle k,z\rangle}$. In case it is, set $j_n = j_{n-1}$. Otherwise, set $j_n = \langle k, y + \ell - 1\rangle$.

Output j_n, and goto Stage $n+1$.

By construction, M is an iterative IIM. We have to show that M infers $\mathcal{C}$. Let $k, z \in \mathbb{N}$, let $L = L_{\langle k,z \rangle}$, and let $t \in pos(L)$. If $\Phi_k(k)$ is undefined, then M outputs in every stage the guess $\langle k, 0 \rangle$. By definition, $L_{\langle k,z \rangle} = L_{\langle k,0 \rangle}$ for all $z \in \mathbb{N}$, and thus, $L = L_{\langle k,0 \rangle}$.

Now assume $\Phi_k(k)$ to be defined. Hence there is a y such that $\Phi_k(k) = y$. By $\mathcal{C}$'s definition, L in particular contains all strings s of form $a^k b a^m$ with $m \geq y$. Since $t \in pos(L)$, M eventually receives a string $s = a^k b a^m$, $m \geq y$, and verifies $\Phi_k(k) \leq m$. Consequently, it rejects its initial guess $\langle k, 0 \rangle$ and changes its mind to $\langle k, y \rangle$. Afterwards, M has to distinguish between finitely many possible candidate hypotheses for L, since $L = L_{\langle k,y+r \rangle}$ for some $r \in \mathbb{N}$ with $r \leq y$. By definition, each of these candidate hypotheses is uniquely characterized by infinitely many strings of form $a^k b^{r+1} a^m$. Thus, if M's actual guess is still incorrect then M eventually receives one of those strings and changes its mind to a correct guess for L. By construction, M repeats this guess in every subsequent stage. Hence, M learns L and Claim 1 is proved.

Claim 2. *No conservative IIM learns $\mathcal{C}$ with respect to $\mathcal{L}$.*

Adapting the proof technique developed in Lange and Zeugmann [12], Claim 2 follows by reducing the halting problem to $\mathcal{C} \in CONSV$ with respect to the hypothesis space $\mathcal{L}$. We omit the details. □

The latter result points to a particular strength of iterative IIMs. That is, iterative learning is not requested to realize the *subset principle* (cf. [21]). Moreover, the proof of the latter theorem shows that *redundancy* in the hypothesis space may lead to a serious increase in the learning power of iterative IIMs. Since $IT \subseteq FB$ as well as $IT \subseteq BEM_k$ for all $k \in \mathbb{N}$, the latter remarks apply to feed-back inference and bounded example memory identification, too. Consequently, one might be tempted to conjecture that even $IT \setminus CONSV \neq \emptyset$. This has also been claimed in Zeugmann and Lange [27] (cf. Theorem 19, Assertion (3)). However, the proof given there is erroneous, and the stated conjecture is definitely false. Having the freedom to take a rich enough suitably chosen hypothesis space does really change the whole picture.

As it turned out, for proving $IT \subseteq CONSV$, $FB \subseteq CONSV$, and $BEM_k \subseteq CONSV$, it is conceptually simpler to use the characterization of conservative learning equating it with set-driven inference (cf. Lange and Zeugmann [13]). Set-drivenness describes the requirement that the output of an IIM is only allowed to depend on the range of its input.

Definition 6 (Wexler and Culicover, Sec. 2.2, [22]). *Let $\mathcal{C}$ be an indexable class.* ***An IIM is said to be set-driven*** *with respect to $\mathcal{C}$ iff its output depends only on the range of its input; that is, iff $M(t_x) = M(\hat{t}_y)$ for all $x, y \in \mathbb{N}$, all positive presentations t, $\hat{t} \in \bigcup_{c \in \mathcal{C}} pos(c)$ provided $t_x^+ = \hat{t}_y^+$.*

Whenever the relevant indexable class $\mathcal{C}$ is clear from the context we refer to set-driven with respect to $\mathcal{C}$ as set-driven for short. By *s-LIM* we denote the collection of all indexable classes that are *LIM*–inferable by some set-driven IIM. Moreover, whenever dealing with set-driven IIMs it is conceptionally advantageous to define or describe them in dependence on the relevant set obtained as input instead of the initial segments of a positive presentation usually fed an IIM.

The next proposition completely characterizes the learning capabilities of set-driven learners (cf. [13]).

Proposition 1. *s-LIM = CONSV.*

Next, we show that every feed-back learner can be simulated by a set-driven IIM.

Theorem 2. *FB ⊆ s-LIM*

Proof. Let $\mathcal{X}$ be the relevant learning domain over which $\mathcal{C}$ is defined, and assume $\mathcal{C} \in FB$. Then there are an IIM M and a hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$ such that M witnesses the feed-back learnability of $\mathcal{C}$ with respect to $\mathcal{H}$. For proving $\mathcal{C} \in s\text{-}LIM$, first we construct a suitable hypothesis space $\hat{\mathcal{H}} = (\hat{h})_{j\in\mathbb{N}}$. Let $\mathcal{F} = F_0, F_1, F_2, \ldots$ denote any repetition free enumeration of all finite subsets of $\mathcal{X}$. Furthermore, we assume an effective procedure computing for every finite set $F \subseteq \mathcal{X}$ its uniquely determined index $\#(F)$ in $\mathcal{F}$. Now we define

$$\hat{h}_j = \begin{cases} h_{\frac{j}{2}}, & \text{if } j \text{ is even,} \\ F_{\frac{j-1}{2}}, & \text{if } j \text{ is odd.} \end{cases}$$

Moreover, for every non-empty finite set $T \subseteq \mathcal{X}$ we define $rf(T) = s_0, s_1, \ldots, s_{\mathrm{card}(T)-1}$ to be the *repetition free* enumeration of all the elements of T in lexicographical order. By $\ell(T)$ we denote the lexicographically largest element of T. Finally, let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be any effective enumeration of all finite sequences of elements from $\mathcal{X}$. The desired set-driven IIM $\hat{M}$ takes as its inputs finite sets T, and is defined as follows:

IIM $\hat{M}$: "On input T do the following:

Test for all $k \leq \mathrm{card}(T)$ whether or not $\sigma_k^+ \subseteq T$. For all k successively passing this test check whether or not the following condition is fulfilled for all $F \subseteq T$.

$$M_{|\sigma_k|}(\sigma_k) = M_{|\sigma_k|+\mathrm{card}(F)}(\sigma_k \cdot rf(F)) = M_{|\sigma_k|+\mathrm{card}(F)+1}(\sigma_k \cdot rf(F) \cdot \ell(F)).$$

If there exists a k passing this test, too, then choose the minimal one, compute $j = M_{|\sigma_k|}(\sigma_k)$, output $2j$, and request the next input.
Otherwise, output $2\#(T)+1$, and request the next input."

By construction, $\hat{M}$ is set-driven und outputs in each learning step a hypothesis. It remains to show that $\hat{M}$ infers $\mathcal{C}$. Let $c \in \mathcal{C}$, and let $t \in pos(c)$. We distinguish the following cases.

Case 1. c is finite.

Then, there exists an $n \in \mathbb{N}$ such that $t_n^+ = c$. Thus, it suffices to show that $c = \hat{h}_{\hat{M}(c)}$. In case $\hat{M}$ outputs $2\#(c)+1$, we are done. Otherwise, $\hat{M}$ has found a finite sequence σ with $\sigma^+ \subseteq c$ that in particular fulfills $M_{|\sigma|}(\sigma) = M_{|\sigma|+m}(\sigma \cdot rf(c)) = M_{|\sigma|+m+1}(\sigma \cdot rf(c) \cdot \ell(c))$, where $m = \mathrm{card}(c)$. Hence, $\hat{M}$ computes $j = M_{|\sigma|}(\sigma)$ and outputs $2j$. Since $\hat{h}_{2j} = h_j$, it suffices to show $c = h_j$. Obviously, $\hat{t} = \sigma \cdot rf(c) \cdot \ell(c), \ell(c), \ldots$ constitutes a positive presentation for c. We know that $j = M_{|\sigma|+m}(\sigma \cdot rf(c)) = M_{|\sigma|+m+1}(\sigma \cdot rf(c) \cdot \ell(c))$. Now, let $Q(j, \ell(c))$ be the query computed by M in Stage $|\sigma| + m + 1$. By construction, this query equals the query computed in Stage $|\sigma| + m + 2$. Since M has already seen all elements belonging to c, it must receive in both stages the same answer. But this implies that M computes in every subsequent stage the query $Q(j, \ell(c))$, too, thereby always receiving the same answer. Consequently, $(M_n(\hat{t}))_{n\in\mathbb{N}}$ converges to j. Since M *FB*–learns c, we have $c = h_j$.

Case 2. c is infinite.

Since M has to learn c from every positive presentation for it, there exists a sequence σ for c such that

$$M_{|\sigma|}(\sigma) = M_{|\sigma|+\mathrm{card}(F)}(\sigma \cdot rf(F)) = M_{|\sigma|+\mathrm{card}(F)+1}(\sigma \cdot rf(F) \cdot \ell(F)) \tag{1}$$

for all subsets $F \subseteq c$. In particular, every locking sequence must fulfil Condition (1) (cf. [16]). For proving $\hat{M}$'s correctness, we fix the sequence σ satisfying (1) that is first enumerated, i.e., let $\sigma = \sigma_{m_0}$, and for all σ_z, $z < m_0$, with $\sigma_z^+ \subseteq c$ we have σ_z does not fulfil (1).

Let t^c be the lexicographically ordered positive presentation of c. Then $t^c_n = rf(t^{c,+}_n)$ for all $n \in \mathbb{N}$. Consequently, $j = M_{|\sigma|}(\sigma) = M_{|\sigma|+n+1}(\sigma \cdot t^c_n)$ for all $n \in \mathbb{N}$. Thus M converges to j when fed $\sigma \cdot t^c$. Moreover, $h_j = c$, since M FB–infers c. Now, it suffices to show that $\hat{M}$ converges to $2j$. Let $t \in pos(c)$ be arbitrarily fixed, and let n_0 be the least index n such that $\sigma^+ \subseteq t^+_n$, and let $m = \max\{n_0, m_0\}$. Hence, on every input t^+_{m+r}, $r \in \mathbb{N}$, $\hat{M}$ finds at least one k fulfilling the tests described. Moreover, since σ_{m_0} is the first enumerated sequence for c satisfying (1), for every other sequence σ_z, $z < m_0$, there must be a finite set $F \subseteq c$ such that

$$M_{|\sigma_z|}(\sigma_z) = M_{|\sigma_z|+\mathrm{card}(F)}(\sigma_z \cdot rf(F)) = M_{|\sigma_z|+\mathrm{card}(F)+1}(\sigma_z \cdot rf(F) \cdot \ell(F))$$

is not fulfilled. Consequently, $\hat{M}$ converges to $2M_{|\sigma|}(\sigma) = 2j$, and we are done. □

The proof technique developed above is also powerful enough to establish the analogous result for bounded example memory inference. This is done in the next theorem.

Theorem 3. *$BEM_m \subseteq$ s-LIM for all $m \in \mathbb{N}$.*

Proof. We use the same notations as in the proof of Theorem 2. In particular, the desired hypothesis space $\hat{\mathcal{H}}$ is defined as above. Now, let $m \in \mathbb{N}$ be arbitrarily fixed. The desired set-driven IIM $\hat{M}$ is essentially defined as in the proof of Theorem 2. However, we have to modify it appropriately to handle the information contained in the example memory. The IIM $\hat{M}$ takes as its inputs finite sets T, and is defined as follows:

IIM $\hat{M}$: "On input T do the following:
Test for all $k \leq \mathrm{card}(T)$ whether or not $\sigma_k^+ \subseteq T$. For all k successively passing this test check whether or not the following condition is fulfilled for all $F \subseteq T$.

$$\begin{aligned}
\pi_1(M_{|\sigma_k|}(\sigma_k)) &= \pi_1(M_{|\sigma_k|+\mathrm{card}(F)}(\sigma_k \cdot rf(F))) \\
&= \pi_1(M_{|\sigma_k|+\mathrm{card}(F)+1}(\sigma_k \cdot rf(F) \cdot \ell(F))) \\
&= \ldots = \pi_1(M_{|\sigma_k|+\mathrm{card}(F)+2m+1}(\sigma_k \cdot rf(F) \cdot \underbrace{\ell(F) \cdot \ldots \cdot \ell(F)}_{(2m+1)\text{ times}}))
\end{aligned}$$

If there exists a k passing this test, too, then choose the minimal one, compute $j = \pi_1(M_{|\sigma_k|}(\sigma_k))$, output $2j$, and request the next input.
Otherwise, output $2\#(T) + 1$, and request the next input."

By construction, $\hat{M}$ is set-driven und outputs in each learning step a hypothesis. It remains to show that $\hat{M}$ infers $\mathcal{C}$. Let $c \in \mathcal{C}$, and let $t \in pos(c)$. We distinguish the following cases.

Case 1. c is finite.

Then, there exists an $n \in \mathbb{N}$ such that $t_n^+ = c$. Thus, it suffices to show that $c = \hat{h}_{\hat{M}(c)}$. In case $\hat{M}$ outputs $2\#(c)+1$, we are done. Otherwise, $\hat{M}$ has found a finite sequence σ with $\sigma^+ \subseteq c$ that in particular fulfills

$$\begin{aligned}\pi_1(M_{|\sigma|}(\sigma)) &= \pi_1(M_{|\sigma|+z}(\sigma \cdot rf(c))) = \pi_1(M_{|\sigma|+z+1}(\sigma \cdot rf(c) \cdot \ell(c))) \\ &= \ldots = \pi_1(M_{|\sigma|+z+2m+1}(\sigma \cdot rf(c) \cdot \ell(c)^{2m+1})),\end{aligned}$$

where $z = \mathrm{card}(c)$. Hence, $\hat{M}$ computes $j = \pi_1(M_{|\sigma|}(\sigma))$ and outputs $2j$. Since $\hat{h}_{2j} = h_j$, it suffices to show $c = h_j$. Obviously, $\hat{t} = \sigma \cdot rf(c) \cdot \ell(c), \ell(c), \ldots$ constitutes a positive presentation for c. We know that

$$\begin{aligned}j = \pi_1(M_{|\sigma|+z}(\sigma \cdot rf(c))) &= \pi_1(M_{|\sigma|+z+1}(\sigma \cdot rf(c) \cdot \ell(c))) \\ &= \ldots = \pi_1(M_{|\sigma|+z+m+1}(\sigma \cdot rf(c) \cdot \ell(c)^{2m+1})). \qquad (2)\end{aligned}$$

We claim that $j = \pi_1(M_{|\sigma|+z+2m+r}(\sigma \cdot rf(c) \cdot \ell(c)^{2m+r}))$ for all $r \geq 1$. Having this claim, we know that M converges on $\hat{t}$, and we are done.

Clearly, M can change its hypothesis only in case it computes and outputs a set S not yet tested. Let $S_{|\sigma|+z}$, $S_{|\sigma|+z+1}$, $\ldots S_{|\sigma|+z+2m+1}$ be the sets output in Stage $|\sigma|+z$, $|\sigma|+z+1$, ..., $|\sigma|+z+2m+1$, respectively. By definition of BEM_m we have $S_{|\sigma|+z+\mu+1} \subseteq S_{|\sigma|+z+\mu} \cup \{\ell(c)\}$ for all $\mu = 0, \ldots, 2m$. Consequently, if M once excludes an element $s \neq \ell(c)$ from the set received as actual input, it cannot include this element again. It may, however, include $\ell(c)$ and exclude it afterwards. Nevertheless, if there are two sets, say $S_{|\sigma|+z+r_1}$ and $S_{|\sigma|+z+r_2}$ such that $S_{|\sigma|+z+r_1} = S_{|\sigma|+z+r_2}$ and $r_1 \neq r_2$, then M will produce a periodic sequence of sets, and Equation 2 implies that the first components of all hypotheses output afterwards are equal, too. Now, it suffices to argue that this event must happen. Since $\mathrm{card}(S_{|\sigma|+z}) \leq m$, M can exclude at most m elements from the set $S_{|\sigma|+z}$ output in Stage $|\sigma|+z$. Additionally, whenever an element is excluded, it may include $\ell(c)$, and it may exclude $\ell(c)$. Hence, the longest sequence of pairwise non-equal sets has length $2m+1$. Finally, since the last element of $rf(c)$ equals $\ell(c)$, the IIM $\hat{M}$ has tested a sequence of length $m+2$, and hence it must have found a period.

Case 2. c is infinite.

Since M has to learn c from every positive presentation for it, there exists a sequence σ for c such that

$$\begin{aligned}\pi_1(M_{|\sigma|}(\sigma)) &= \pi_1(M_{|\sigma|+\mathrm{card}(F)}(\sigma \cdot rf(F))) = \pi_1(M_{|\sigma|+\mathrm{card}(F)+1}(\sigma \cdot rf(F) \cdot \ell(F))) \\ &= \ldots = \pi_1(M_{|\sigma|+\mathrm{card}(F)+2m+1}(\sigma \cdot rf(F) \cdot \ell(F)^{2m+1})) \qquad (3)\end{aligned}$$

for all subsets $F \subseteq c$. In particular, every locking sequence must fulfil Condition (3) (cf. [16]). For proving $\hat{M}$'s correctness, we fix the sequence σ satisfying (3) that is first enumerated, i.e., let $\sigma = \sigma_{r_0}$, and for all σ_z, $z < r_0$, with $\sigma_z^+ \subseteq c$ we have that σ_z does not fulfil Condition (3).

Let t^c be the lexicographically ordered positive presentation of c. Then $t^c_n = rf(t^{c,+}_n)$ for all $n \in \mathbb{N}$. Consequently, $j = \pi_1(M_{|\sigma|}(\sigma)) = \pi_1(M_{|\sigma|+n+1}(\sigma \cdot t^c_n))$ for all $n \in \mathbb{N}$. Thus M converges to j when fed $\sigma \cdot t^c$. Moreover, $h_j = c$, since M BEM_m–infers c. Now, it suffices to show that $\hat{M}$ converges to $2j$. Let $t \in pos(c)$ be arbitrarily fixed, let n_0 be the least index n such that $\sigma^+ \subseteq t^+_n$, and let $r = \max\{n_0, r_0\}$. Therefore, on every input $t^+_{r+\rho}$, $\rho \in \mathbb{N}$, $\hat{M}$ finds at least one k fulfilling the tests described. Moreover, since σ_{r_0} is the first enumerated sequence for c satisfying (3), for every other sequence σ_z, $z < r_0$, there must be a finite set $F \subseteq c$ such that

$$\begin{aligned} \pi_1(M_{|\sigma|}(\sigma)) &= \pi_1(M_{|\sigma|+\mathrm{card}(F)}(\sigma \cdot rf(F))) = \pi_1(M_{|\sigma|+\mathrm{card}(F)+1}(\sigma \cdot rf(F) \cdot \ell(F))) \\ &= \ldots = \pi_1(M_{|\sigma|+\mathrm{card}(F)+2m+1}(\sigma \cdot rf(F) \cdot \ell(F)^{2m+1})) \end{aligned}$$

is not fulfilled. Consequently, $\hat{M}$ converges to $2M_{|\sigma|}(\sigma) = 2j$, and we are done. □

The latter theorems immediately allow the corollary that every iterative IIM can be simulated by a set-driven IIM, too. Nevertheless, we present a separate proof for it, since the construction considerably simplifies. Note that Kinber and Stephan [6] also proved this result in the setting of learning recursively enumerable languages.

Corollary 4. *IT* $\subseteq$ *s-LIM*

Proof. We use the same notations as in the proof of Theorem 2. Then, the desired set-driven IIM $\hat{M}$ takes as its inputs finite sets T, and is defined as follows:

IIM $\hat{M}$: "On input T do the following: Compute $rf(T)$, and $\ell(T)$. Check whether or not $M_{\mathrm{card(T)}-1}(rf(T)) = M_{\mathrm{card(T)}}(rf(T) \cdot \ell(T))$.
If it is, output $2M_{\mathrm{card(T)}-1}(rf(T))$, and request the next input.
Otherwise, output $2\#(T)+1$, and request the next input."

By definition, $\hat{M}$ is set-driven. It remains to show that $\hat{M}$ *LIM*–infers $\mathcal{C}$ with respect to $\hat{\mathcal{H}}$. Let $c \in \mathcal{C}$, and let $t \in pos(c)$. We distinguish the following cases.

Case 1. c is finite.

If $\hat{M}(c) = 2\#(c)+1$ we are done by construction. Otherwise, $M_{\mathrm{card(c)}-1}(rf(c)) = M_{\mathrm{card(c)}}(rf(T) \cdot \ell(c))$. Let $j = M_{\mathrm{card(c)}-1}(rf(c))$; then rewriting $M_{\mathrm{card(c)}}(rf(c) \cdot \ell(c))$ yields $M_{\mathrm{card(c)}}(rf(c) \cdot \ell(c)) = M(j, \ell(c)) = M(j, \ell(c)) = M_{\mathrm{card(c)}+1}(rf(T) \cdot \ell(c))$. Hence, M converges on the positive presentation $rf(c) \cdot \ell(c)^\infty \in pos(c)$. Since M learns c, we are again done.

Case 2. c is infinite.

Let $t^c = w_0,\ w_1,\ w_2,\ \ldots$ be the lexicographically ordered positive presentation of c. Since M iteratively learns c from t^c, there exists an $n_0 \in \mathbb{N}$ such that $M_{n_0}(t^c) = M_n(t^c)$ for all $n \geq n_0$. Therefore, we may conclude that $M_{n_0+1}(t^c_{n_0} \cdot w) = M_{n_0}(t^c)$ for all $w \in c \setminus t^{ord,+}_{n_0}$. For seeing this, let $n_0 + z$ be the uniquely determined index of w in t^c, i.e., $w = w_{n_0+z}$. Letting $j = M_{n_0}(t^c)$ we have

$$j = M_{n_0+1}(t^c) = M(j, w_{n_0+1}) = \ldots = M(j, w_{n_0+z}) = M_{n_0+z}(t^c).$$

Consequently, there is an initial segment $\sigma = t^c_{n_0}$ of the lexicographically ordered positive presentation of c on which M is locked provided this segment is extended with elements

$w \in c \setminus t^{ord,+}_{n_0}$. Since M learns c from t^c we know that $c = h_j = \hat{h}_{2j}$. Finally, since $t \in pos(c)$ there exists an index m_0 such that $\sigma^+ \subseteq t^+_{m_0}$. Thus, σ is a prefix of $rf(t^+_{m_0})$, and hence $\hat{M}(t^+_m) = 2j$ for all $m \geq m_0$. □

We finish this subsection by proving the following upper bound for the learning capabilities of iterative learning, feed-back inference, and bounded example memory identification.

Theorem 5.

(1) $IT \subset CONSV$,

(2) $FB \subset CONSV$,

(3) $\bigcup_{k \in \mathbb{N}} BEM_k \subset CONSV$.

Proof. Since $IT \subseteq FB$ as well $IT \subseteq BEM_k$ for every $k \in \mathbb{N}$, Assertion (1) directly follows from (2) or (3). Furthermore, by Theorems 2 and 3 as well as Corollary 4 we obtain from Proposition 1 the containment of FB, BEM_k and IT in $CONSV$, respectively. Thus, it suffices to prove that the stated inclusions are proper.

Claim 1. $CONSV \setminus FB \neq \emptyset$.

Let $L = \{a\}^+$, and for all $j,\ k \in \mathbb{N}$ let $L_{j,k} = \{a^m \mid\ 1 \leq m \leq j+1\} \cup \{b^{j+1}, a^{j+1+k}\}$. Let $\mathcal{C}_{FB}$ and $\mathcal{H}$ be the collection and canonical enumeration of all the languages $L_{j,k}$ and L, respectively. One straightforwardly verifies $\mathcal{C}_{FB} \in CONSV$ with respect to $\mathcal{H}$.

Next, we show $\mathcal{C}_{FB} \notin FB$. Suppose there is an IIM M which FB–learns $\mathcal{C}_{FB}$. Let $\sigma = s_0,\ \ldots,\ s_n$ be a locking sequence for $L = \{a\}^+$ (cf. [16]). Furthermore, let $m = max\{z \mid\ a^z \in \sigma^+\}$, and let $j_n = M_n(\sigma)$. Thus, when fed any finite extension of σ with strings from L, M repeats its guess j_n.

Now we select two different finite languages $\hat{L}$ and $\tilde{L}$ from $\mathcal{C}_{FB}$ and show that M fails to learn at least one of them. Let $\hat{L} = \{a^z \mid\ 1 \leq z \leq m\} \cup \{b^m\}$, and consider M's behavior when successively fed the following text $\hat{t} \in pos(\hat{L})$. Let $\hat{t} = \sigma \cdot a,\ \ldots,\ a^m,\ b^m,\ a,\ a,\ \ldots$. Since M learns $\hat{L}$ from $\hat{t}$, there is a $z > n+m+1$ such that M, after processing the initial segment $\hat{t}_z$, does not perform any further mind change. By definition, $\hat{t}_{y+1} = \hat{t}_y, \cdot a$ for all $y \geq z$. Consequently, after M has processed the initial segment $\hat{t}_z$ it must always ask the same question. Thus, M asks at most finitely many different questions when fed $\hat{t}$. Now, choose any string $\tilde{s} \in L \setminus \hat{L}$ the IIM M is never asking for, and set $\tilde{L} = \hat{L} \cup \{\tilde{s}\}$.

Finally, let $\tilde{t} = \sigma \cdot \tilde{s},\ a,\ \ldots,\ a^m,\ b^m,\ a,\ a,\ \ldots$. Thus $\tilde{t} \in pos(\tilde{L})$. Since σ is a locking sequence for L, M outputs j_n after processing the initial segment $\sigma \cdot \tilde{s},\ a,\ \ldots,\ a^m$. Afterwards, M receives the string b^m. Based on the local input j_n and b^m, M must ask the same question, say s', as in case the initial segment $\sigma \cdot a,\ \ldots,\ a^m,\ b^m$ has been processed. Since $s' \neq \tilde{s}$, M obtains the same answer and generates, therefore, the same guess as in the former case. Now, the same argument may be iterated in order to show that M, when successively fed $\tilde{t}$, generates afterwards the same sequence of hypotheses as in case that M is processing $\hat{t}$. Hence, M is fooled.

Claim 2. $CONSV \setminus \bigcup_{k \in \mathbb{N}} BEM_k \neq \emptyset$.

For all $j \in \mathbb{N}$, let $L_j = \{a\}^+ \setminus \{a^{j+1}\}$. Let $\mathcal{C}_{CONSV}$ and $\mathcal{H} = (L_j)_{j \in \mathbb{N}}$ be the collection and canonical enumeration of all these languages L_j, respectively. Again, $\mathcal{C}_{CONSV} \in CONSV$ can be easily verified.

It remains to show that $\mathcal{C}_{CONSV} \notin \bigcup_{k \in \mathbb{N}} BEM_k$. Suppose the converse, i.e., there are a $k \in \mathbb{N}$, an IIM M, and a hypothesis space $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ such that M BEM_k–infers $\mathcal{C}_{CONSV}$ with respect to $\mathcal{H}$. Since M has to infer $L_0 = \{a\}^+ \setminus \{a\}$, there has to be a locking sequence σ for L_0 (cf. [16]). Hence, we know that, $\pi_1(M_{|\sigma|}(\sigma)) = \pi_1(M_{|\sigma|+r}(\sigma \cdot t))$ for all $t \in pos(L_0)$ and all $r \in \mathbb{N}$.

The following lemma is telling us that the IIM M, while attempting to learn $\mathcal{C}_{CONSV}$, must almost always include the actual string into its new set of stored examples.

Lemma 1. *Let $t \in pos(L_0)$ be any positive presentation starting with the locking sequence σ, and let $y \geq |\sigma|$. Then, $s \in \pi_2(M_{y+1}(t_y \cdot s))$ for almost all $s \in L_0$.*

Suppose the converse, i.e., there exist infinitely many strings $s \in L_0$ that satisfy $s \notin \pi_2(M_{y+1}(t_y \cdot s))$. By definition of BEM_k, both $\pi_2(M_{y+1}(t_y \cdot s)) \subseteq \pi_2(M_y(t_y)) \cup \{s\}$ and $\mathrm{card}(\pi_2(M_{y+1}(t_y \cdot s))) \leq k$ have to be valid. Since there are at most 2^k possible subsets of $\pi_2(M_y(t_y))$, we can effectively find two different strings u, $v \in L_0$ with u, $v \notin t_y^+$ satisfying $\pi_2(M_{y+1}(t_y \cdot u)) = \pi_2(M_{y+1}(t_y \cdot v))$. Moreover, we know that $j = \pi_1(M_{y+1}(t_y \cdot u)) = \pi_1(M_{y+1}(t_y \cdot v))$, since σ is a locking sequence for L_0. Now, set $L_u = \{a\}^+ \setminus \{u\}$ and $L_v = \{a\}^+ \setminus \{v\}$, and let t' be any positive presentation of $L' = \{a\}^+ \setminus \{u, v\}$. Consider M's behavior when fed one of the following texts $t^u = t_y \cdot v \cdot t'$ and $t^v = t_y \cdot u \cdot t'$ for L_u and L_v, respectively. Since $t^u_y = t^v_y$, we get $M_z(t^u) = M_z(t^v)$ for all $z \leq y$. By construction, $M_{y+1}(t^u) = M_{y+1}(t^v)$. Past point $y+1$, both texts are identical, and hence $M_{y+z}(t^u) = M_{y+z}(t^v)$ for all $z \geq 0$, too. Consequently, M produces on both texts the same sequence of hypotheses, a contradiction. Hence, Lemma 1 is shown.

Now, Lemma 1 may be used to obtain the following insight:

Lemma 2. *There are two finite sequences τ and ρ of strings from $L_0 \setminus \sigma^+$ with $\tau^+ \# \rho^+$ as well as a string $s \in L_0 \setminus (\sigma^+ \cup \tau^+ \cup \rho^+)$ such that $M_{|\sigma|+|\tau|+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+|\rho|+1}(\sigma \cdot \rho \cdot s)$.*

Choose k pairwise disjoint sets W_0, ..., W_{k-2}, W_{k-1} such that, for all $0 \leq i \leq k-1$, $W_i \subseteq L_0 \setminus \sigma^+$ and $\mathrm{card}(W_i) = m = (4e)^{k-1}$ are fulfilled. By FS we denote the set of all finite sequences $\xi = w_0$, ..., w_{k-1} with $w_0 \in W_0$, ..., and $w_{k-1} \in W_{k-1}$, respectively. By construction, $\xi_1^+ \# \xi_2^+$ for all $\xi_1, \xi_2 \in FS$ provided $\xi_1 \neq \xi_2$. Given any $\xi \in FS$; we know by Lemma 1 that $s' \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s'))$ for almost all $s' \in L_0$. Hence, we can effectively find a string $s \in L_0 \setminus (\sigma^+ \cup \bigcup_{i<k} W_i)$ that satisfies $s \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all $\xi \in FS$. Next, we show that there must be two finite sequences τ, $\rho \in FS$ with $M_{|\sigma|+k+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+k+1}(\sigma \cdot \rho \cdot s)$.

Since σ is a locking sequence for L_0, we know that $\pi_1(M_{|\sigma|}(\sigma)) = \pi_1(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all $\xi \in FS$. Hence, it suffices to estimate the maximal number $\hat{N}$ of pairwise different sets of cardinality at most k that may be output by M after having fed $\sigma \cdot \xi \cdot s$, where ξ ranges over FS. Let $S = \pi_2(M_{|\sigma|}(\sigma))$; and assume the worst case, that is, $\mathrm{card}(S) = k$. By N_0, N_1, ..., N_k we denote the number of possible sets of cardinality 0, 1 ..., k, respectively, M may output after having been fed $\sigma \cdot \xi \cdot s$, where ξ ranges over FS. By construction we know that $s \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all $\xi \in FS$. Hence, $N_0 = 0$, and $N_1 = 1$. In general, a set of cardinality j can contain ℓ elements maintained from S, and

$j-1-\ell$ elements stored while processing a sequence $\xi \in FS$, where $0 \leq \ell \leq j-1$, and it must contain s. Thus, we obtain:

$$N_j = \sum_{\ell=0}^{j-1} \binom{k}{\ell} \binom{k}{j-1-\ell} m^{j-1-\ell} \tag{4}$$

Since $\mathrm{card}(\pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))) \leq k$, we get from (4):

$$\begin{aligned}
\hat{N} &= \sum_{j=0}^{k} N_j = 0 + 1 + \sum_{j=2}^{k} N_j \\
&= 1 + \sum_{j=2}^{k} \sum_{\ell=0}^{j-1} \binom{k}{\ell} \binom{k}{j-1-\ell} m^{j-1-\ell} \\
&= \sum_{j=1}^{k} \sum_{\ell=0}^{j-1} \binom{k}{\ell} \binom{k}{j-1-\ell} m^{j-1-\ell} \\
&= \sum_{j=0}^{k-1} \sum_{\ell=0}^{j} \binom{k}{\ell} \binom{k}{j-\ell} m^{j-\ell} \qquad (5)
\end{aligned}$$

Next, we estimate $\hat{N}$ by replacing $m^{j-\ell}$ by m^j, and by applying Vandermonde's convolution (cf. [8]). Thus, (5) yields:

$$\begin{aligned}
\hat{N} &< \sum_{j=0}^{k-1} m^j \sum_{\ell=0}^{j} \binom{k}{\ell} \binom{k}{j-\ell} \\
&< \sum_{j=0}^{k-1} m^j \sum_{\ell} \binom{k}{\ell} \binom{k}{j-\ell} \\
&= \sum_{j=0}^{k-1} m^j \binom{2k}{j} \qquad (6)
\end{aligned}$$

Finally, we estimate m^j by m^{k-1}. This leaves essentially a partial sum of the $2k$th row in Pascal's triangle. Unfortunately, there is no closed form for such partials sums. Therefore, we apply the well-known estimate

$$\sum_{j=0}^{d} \binom{n}{j} \leq \left(\frac{en}{d}\right)^d \text{ for all } n \geq d \geq 1 \tag{7}$$

Putting it all together, we obtain from (6) and (7):

$$\begin{aligned}
\hat{N} &< m^{k-1} \sum_{j=0}^{k-1} \binom{2k}{j} \leq m^{k-1} \left(\frac{2ke}{k-1}\right)^{k-1} \\
&\leq m^{k-1} (4e)^{k-1} \qquad (8)
\end{aligned}$$

Since $\mathrm{card}(FS) = m^k = ((4e)^{k-1})^k = m^{k-1}(4e)^{k-1}$ Inequality (8) tells us that there are more sequences in FS than sets M may possibly output. Thus, there must exist τ and $\rho \in FS$ with $\tau^+ \# \rho^+$ such that $M_{|\sigma|+|\tau|+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+|\rho|+1}(\sigma \cdot \rho \cdot s)$. This proves Lemma 2.

Now we are ready for showing that M fails to identify $\mathcal{C}_{CONSV}$. We define two languages $\hat{L}$, $\tilde{L} \in \mathcal{C}_{CONSV}$ witnessing M's weakness. Select τ, ρ, and s in a way such that the requirements of Lemma 2 are fulfilled. By assumption, $\tau^+ \setminus \rho^+ \neq \emptyset$ as well as $\rho^+ \setminus \tau^+ \neq \emptyset$. Select any $\hat{w} \in \rho^+ \setminus \tau^+$ and any $\tilde{w} \in \tau^+ \setminus \rho^+$. Set $\hat{L} = L_0 \setminus \{\hat{w}\}$ and $\tilde{L} = L_0 \setminus \{\tilde{w}\}$. Let t' be any positive presentation of $L_0 \setminus \{\hat{w}, \tilde{w}\}$. Since $(\tau^+ \cup \rho^+) \cap \sigma^+ = \emptyset$, $\hat{w} \in \rho^+ \setminus \tau^+$ and $s \notin \rho^+$, we may conclude that $\hat{w} \neq s$ and $\hat{w} \notin \sigma^+ \cup \tau^+$. Thus, $\hat{t} = \sigma \cdot \tau \cdot s \cdot t'$ defines a positive presentation of $\hat{L}$. Similarly, it can be easily verified that $\tilde{t} = \sigma \cdot \rho \cdot s \cdot t'$ belongs to $pos(\tilde{L})$. By Lemma 2, $M_{n+k+1}(\hat{t}) = M_{n+k+1}(\tilde{t})$. Past point $n+k+1$, both texts are identical, and hence $M_{n+k+1+z}(\hat{t}) = M_{n+k+1+z}(\tilde{t})$ for all $z \geq 0$, too. Consequently, M fails to learn $\hat{L}$ or $\tilde{L}$ when fed $\hat{t}$ and $\tilde{t}$, respectively, a contradiction. This finishes the proof of Claim 2. □

3.2. On the Strength of Incremental Learning

Now we study to what extend, if ever, feed-back learning and bounded example memory inference, respectively, enlarges the learning capabilities of iterative learning. Interestingly, even the ability to store exactly one distinguished example seriously increases the learning capabilities of iterative IIMs. Moreover, the ability to ask whether or not a particular element did already appear in the initial segment of the positive presentation processed so far considerably increases the learning capabilities of iterative IIMs, too.

Theorem 6.

(1) $IT \subset FB$,

(2) $IT \subset BEM_1$,

(3) $BEM_k \subset BEM_{k+1}$ *for all* $k \in \mathbb{N}$.

Proof. Since $IT \subseteq FB$, $IT = BEM_0$, and $BEM_k \subseteq BEM_{k+1}$ for all $k \in \mathbb{N}$, it suffices to show that all the stated inclusions are proper.

Claim 1. $FB \setminus IT \neq \emptyset$.

The indexable class $\mathcal{C}_{CONSV}$ used in the proof of Theorem 5 witnesses the desired separation, too. Since $\mathcal{C}_{CONSV} \notin \bigcup_{k \in \mathbb{N}} BEM_k$, it suffices to prove $\mathcal{C}_{CONSV} \in FB$. Let $\mathcal{H} = (L_j)_{j \in \mathbb{N}}$, where $L_j = \{a\}^+ \setminus \{a^{j+1}\}$. Let $L \in \mathcal{C}_{CONSV}$, and let $t = (s_j)_{j \in \mathbb{N}} \in pos(L)$. The desired feed-back IIM M is defined in stages, where Stage n conceptually describes M_n.

Stage 0. On input s_0 do the following.
Set $j_0 = 0$, output j_0, and goto Stage 1.

Stage n, $n \geq 1$. On input j_{n-1} and s_n do the following.
Make the query '$a^{1+j_{n-1}}$.' If the answer is 'NO,' set $j_n = j_{n-1}$, output j_n, and goto Stage $n+1$.
If the answer is 'YES,' set $j_n = j_{n-1} + 1$, output j_n, and goto Stage $n+1$.

Let $L = L_z = \{a\}^+ \setminus \{a^{z+1}\}$. Since t is a text for L, there is a least $n \in \mathbb{N}$ such that the initial segment t_n contains all strings a^m with $1 \leq m \leq z$. By definition, M asks the oracle in Stage $n+z$ whether or not the string a^{z+1} has already been presented. Since $a^{z+1} \notin L$, the reply is 'NO'. Thus, M repeats this query in every subsequent stage. Finally, the oracle's reply enables M to output the correct guess z. Hence, M FB–learns L from t.

Claim 2. $BEM_1 \setminus IT \neq \emptyset$.

Recall the definition of the indexable class $\mathcal{C}_{FB}$ introduced in the demonstration of Theorem 5, Claim 1. That is, $\mathcal{C}_{FB}$ is the collection consisting of the language $L = \{a\}^+$ and of all the languages $L_{j,k} = \{a^m \mid 1 \leq m \leq j+1\} \cup \{b^{j+1}, a^{j+1+k}\}$. We set $\mathcal{C}_{BEM_1} = \mathcal{C}_{FB}$, and show that $\mathcal{C}_{BEM_1}$ witnesses the desired separation. Since $\mathcal{C}_{BEM_1} \notin FB$ (cf. the demonstration of Theorem 5, Claim 1) and $IT \subseteq FB$, it suffices to show that $\mathcal{C}_{BEM_1} \in BEM_1$.

Let $\mathcal{H}_{BEM_1} = (h_j)_{j \in \mathbb{N}}$ be the canonical enumeration of all the languages in $\mathcal{C}_{BEM_1}$. We define an IIM M which BEM_1–identifies $\mathcal{C}_{BEM_1}$ with respect to $\mathcal{H}_{BEM_1}$. Let $L' \in \mathcal{C}_{BEM_1}$, and let $t = (s_j)_{j \in \mathbb{N}} \in pos(L')$. The IIM M is defined in stages, where Stage n conceptually describes M_n.

Stage 0. On input s_0 do the following.

If $s_0 = a^{k+1}$ for some $k \in \mathbb{N}$, then set $S_0 = \{a^{k+1}\}$ and compute the canonical index z for $L = \{a\}^+$.

Otherwise, i.e., $s_0 = b^{k+1}$ for some $k \in \mathbb{N}$, set $S_0 = \emptyset$, and determine the canonical index z for $L_{k,0} = \{a^m \mid 1 \leq m \leq k+1\} \cup \{b^{k+1}\}$.

Set $j_0 = z$, output $\langle j_0, S_0 \rangle$, and goto Stage 1.

Stage n, $n \geq 1$. On input $\langle j_{n-1}, S_{n-1} \rangle$ and s_n do the following.

If $s_n = b^{k+1}$ for some $k \in \mathbb{N}$, then determine the canonical index z for the language $L_{k,0} \cup S_{n-1}$. Set $j_n = z$, $S_n = \emptyset$, output $\langle j_n, S_n \rangle$, and goto Stage $n+1$.

If $s_n = a^{k+1}$ for some $k \in \mathbb{N}$, test whether $S_{n-1} = \emptyset$. If it is, then determine the canonical index z for the language $h_{n-1} \cup \{a^{k+1}\}$, and set $S_n = \emptyset$ and $j_n = z$.

If $S_{n-1} \neq \emptyset$ then set $j_n = j_{n-1}$. Let $S_{n-1} = \{s\}$; check whether $|s_n| \geq |s|$. If it is, set $S_n = \{s_n\}$. Otherwise, let $S_n = \{s\}$.

Output $\langle j_n, S_n \rangle$, and goto Stage $n+1$.

It remains to show that M infers L'. Obviously, if $L' = L$, then M outputs in every stage a correct guess for L'. Now, let $L' = L_{j,k}$ for some j, $k \in \mathbb{N}$. Hence, there must be an n such that $s_n = b^{j+1}$. By construction, S_{n-1} contains the longest string of form a^{m+1}, say s, which has been presented so far. If $s = a^{j+k+1}$ or $L' = L_{j,0}$, then M outputs in this and every subsequent stage a correct guess for L'. Otherwise, M guesses the finite sublanguage $L_{j,0}$ of L' in Stage n. Since $t \in pos(L')$, M eventually receives in some subsequent stage the string a^{j+k+1}. Then it changes its mind to a correct guess for L' that is repeated in every subsequent stage. Thus, M BEM_1–learns L', and Claim 2 is shown.

Claim 3. $BEM_{k+1} \setminus BEM_k \neq \emptyset$ *for all* $k \in \mathbb{N}$.

Because of $BEM_0 = IT$, the $k = 0$ case has been already shown (cf. Claim 2). The general case is handled by enlarging the indexable class $\mathcal{C}_{BEM_1}$. Let $k \in \mathbb{N}^+$, and let $\langle \ell_0, \ldots, \ell_k \rangle$ be any $k+1$-tuple of natural numbers. For every $j \in N$ let $L_{\langle j, \ell_0, \ldots, \ell_k \rangle} = \{a^m \mid 1 \leq m \leq j+1\} \cup \{b^{j+1}, a^{\ell_0}, \ldots, a^{\ell_k}\}$, and let $L = \{a\}^+$. By $\mathcal{C}_{BEM_{k+1}}$ and $\mathcal{H}_{BEM_{k+1}}$

we denote the collection and canonical enumeration of all the languages $L_{\langle j,\ell_0,\ldots,\ell_k\rangle}$ and L, respectively. By definition, $\mathcal{C}_{BEM_1} \subseteq \mathcal{C}_{BEM_{k+1}}$.

$\mathcal{C}_{BEM_{k+1}} \in BEM_{k+1}$ with respect to $\mathcal{H}_{BEM_{k+1}}$ can be shown using a minor modification of the IIM M defined in Claim 2 above. As long as no string of the form b^m occurs, the modified IIM $\hat{M}$ simply stores the $(k+1)$st longest strings of the form a^n seen so far, and outputs the canonical index for $\{a\}^+$ along with this set. If a string w of the form b^m appears, M outputs the canonical index for the least language L' which contains both w and the $(k+1)$st longest string from L seen so far. Past that point, there is no need to store any further string, since the target language has to be a finite superset of L', and moreover, in case that L' does not equal the target language, the missing strings have to appear in some subsequent step. We omit further details.

The remaining part, i.e., $\mathcal{C}_{BEM_{k+1}} \notin BEM_k$, is harder to prove. Suppose the converse, i.e., there is a hypothesis space $\mathcal{H}$ and a BEM_k IIM M inferring $\mathcal{C}_{BEM_{k+1}}$ with respect to $\mathcal{H}$. Since M learns $L = \{a\}^+$, there has to be a locking sequence σ for L (cf. [16]). Hence, we know that, $j = \pi_1(M_{|\sigma|}(\sigma) = \pi_1(M_{|\sigma|+r}(\sigma \cdot t))$ for all $t \in pos(L)$ and all $r \in \mathbb{N}$.

The following lemma is telling us that the IIM M, while attempting to infer $\mathcal{C}_{BEM_{k+1}}$, must almost always include the actual string into its new set of stored examples. Interestingly, if M violates this constraint it would even fail to learn the proper subclass $\mathcal{C}_{BEM_1}$.

Lemma 3. *Let $t \in pos(L)$ be any positive presentation starting with the locking sequence σ, and let $y \geq |\sigma|$. Then, $s \in \pi_2(M_{y+1}(t_y \cdot s))$ for almost all $s \in L$.*

Suppose the converse, i.e., there exist infinitely many strings $s \in L$ satisfying $s \notin \pi_2(M_{y+1}(t_y \cdot s))$. By definition of BEM_k, we know that both $\pi_2(M_{y+1}(t_y \cdot s)) \subseteq \pi_2(M_y(t_y))$ and $\mathrm{card}(\pi_2(M_{y+1}(t_y \cdot s))) \leq k$ have to be valid. Since there are at most 2^k possible subsets of $\pi_2(M_y(t_y))$, we can effectively find two different strings $u,\ v \in L$ with $|u|,\ |v| > m = \max\{j \mid a^j \in t_y^+\}$ that satisfy $\pi_2(M_{y+1}(t_y \cdot u)) = \pi_2(M_{y+1}(t_y \cdot v))$. Furthermore, $j = \pi_1(M_{y+1}(t_y \cdot u)) = \pi_1(M_{y+1}(t_y \cdot v))$, since σ is a locking sequence for L and $t_y^+ \cup \{u, v\} \subseteq L$. Now, set $L' = \{a^r \mid 1 \leq r \leq m\}$, $L_u = L' \cup \{b^m, u\}$ and $L_v = L' \cup \{b^m, v\}$, and let t' be any positive presentation of L'. Consider M's behavior when fed one of the following texts $t^u = t_y \cdot u \cdot b^m \cdot t'$ and $t^v = t_y \cdot v \cdot b^m \cdot t'$ for L_u and L_v, respectively. Since $t_y^u = t_y^v$, we get $M_z(t^u) = M_z(t^v)$ for all $z \leq y$. By construction, $M_{y+1}(t^u) = M_{y+1}(t^v)$. Past point $y+1$, both texts are identical, and hence $M_{y+z}(t^u) = M_{y+z}(t^v)$ for all $z \geq 0$, too. Consequently, M produces on both texts the same sequence of hypotheses, a contradiction, and Lemma 3 is shown.

Let $m = \max\{j \mid a^j \in \sigma^+\}$, and set $L' = \{a^r \mid 1 \leq r \leq m\}$. Similarly as in the demonstration of Theorem 5, we may use Lemma 3 to arrive at the following conclusion:

Lemma 4. *There are two finite sequences τ and ρ of strings from $L \setminus L'$ with $\tau^+ \# \rho^+$ and a string $s \in L \setminus (L' \cup \tau^+ \cup \rho^+)$ such that $M_{|\sigma|+|\tau|+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+|\rho|+1}(\sigma \cdot \rho \cdot s)$.*

Select k pairwise disjoint sets $W_0, \ldots, W_{k-2}, W_{k-1}$ such that, for all $0 \leq i \leq k-1$, $W_i \subseteq L \setminus L'$ and $\mathrm{card}(W_i) = m = (4e)^{k-1}$ are fulfilled. Let FS be the set of all finite sequences $\xi = w_0, \ldots, w_{k-1}$ with $w_0 \in W_0, \ldots,$ and $w_{k-1} \in W_{k-1}$, respectively. By construction, $\xi_1^+ \# \xi_2^+$ for all $\xi_1, \xi_2 \in FS$ provided $\xi_1 \neq \xi_2$. Given any $\xi \in FS$; Lemma 3 results in $s' \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s'))$ for almost all $s' \in L$. Consequently, we can effectively determine a string $s \in L \setminus (L' \cup \bigcup_{i<k} W_i)$ that satisfies $s \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all

$\xi \in FS$. Now, we show that there must be two finite sequences τ, $\rho \in FS$ that satisfy $M_{|\sigma|+k+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+k+1}(\sigma \cdot \rho \cdot s)$.

Since σ is a locking sequence for L, we know that $\pi_1(M_{|\sigma|}(\sigma)) = \pi_1(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all $\xi \in FS$. Hence, it suffices to estimate the maximal number $\hat{N}$ of pairwise different sets of cardinality at most k that may be output by M after having fed $\sigma \cdot \xi \cdot s$, where ξ ranges over FS. Let $S = \pi_2(M_{|\sigma|}(\sigma))$; and assume the worst case, that is, $\mathrm{card}(S) = k$. By N_0, N_1, ..., N_k we denote the number of possible sets of cardinality 0, 1 ..., k, respectively, M may output after having been fed $\sigma \cdot \xi \cdot s$, where ξ ranges over FS. By construction we know that $s \in \pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))$ for all $\xi \in FS$. Hence, $N_0 = 0$, and $N_1 = 1$. In general, a set of cardinality j can contain ℓ elements maintained from S, and $j-1-\ell$ elements stored while processing a sequence $\xi \in FS$, where $0 \leq \ell \leq j-1$, and it must contain s. Thus, we obtain:

$$N_j = \sum_{\ell=0}^{j-1} \binom{k}{\ell} \binom{k}{j-1-\ell} m^{j-1-\ell} \tag{9}$$

Together with $\mathrm{card}(\pi_2(M_{|\sigma|+k+1}(\sigma \cdot \xi \cdot s))) \leq k$ we get from (9):

$$\hat{N} \quad = \quad \sum_{j=0}^{k} N_j \tag{10}$$

Finally, the same calculation as in the demonstration of Theorem 5, Lemma 2 gives us the following bound:

$$\hat{N} \quad < m^{k-1}(4e)^{k-1} \tag{11}$$

Since $\mathrm{card}(FS) = m^k = ((4e)^{k-1})^k = m^{k-1}(4e)^{k-1}$, Inequality (11) tells us that there are more sequences in FS than sets M may possibly output. Thus, there must exist τ and $\rho \in FS$ with $\tau^+ \# \rho^+$ such that $M_{|\sigma|+|\tau|+1}(\sigma \cdot \tau \cdot s) = M_{|\sigma|+|\rho|+1}(\sigma \cdot \rho \cdot s)$. This proves Lemma 4.

Now we are ready for showing that M is not able to identify $\mathcal{C}_{BEM_{k+1}}$. We fix two languages $\hat{L}$, $\tilde{L} \in \mathcal{C}_{BEM_{k+1}}$ witnessing M's weakness. Choose τ, ρ, and s in accordance with Lemma 4. Note that $k = |\tau| = |\rho|$ as well as $k = \mathrm{card}(\tau^+) = \mathrm{card}(\rho^+)$. Set $\hat{L} = L' \cup \{b^m, s\} \cup \tau^+$ and $\tilde{L} = L' \cup \{b^m, s\} \cup \rho^+$. Obviously, $\hat{L}$, $\tilde{L} \in \mathcal{C}_{BEM_{k+1}}$, and, because of $\tau^+ \# \rho^+$ as well as $L' \cap (\tau^+ \cup \rho^+) = \emptyset$ we have $\hat{L} \neq \tilde{L}$. Let t' be any positive presentation of L', and consider M's behavior when fed one of the following texts $\hat{t} = \sigma \cdot \tau \cdot s \cdot b^m \cdot t'$ and $\tilde{t} = \sigma \cdot \rho \cdot s \cdot b^m \cdot t'$ for $\hat{L}$ and $\tilde{L}$, respectively. By Lemma 4, we know that $M_{|\sigma|+k+1}(\hat{t})$ equals $M_{|\sigma|+k+1}(\tilde{t})$. Past point $|\sigma|+k+1$, both texts are identical. Hence by definition of BEM_k, we may conclude that $M_{|\sigma|+k+1+z}(\hat{t}) = M_{|\sigma|+k+1+z}(\tilde{t})$ for all $z \geq 0$, too. Consequently, M, if ever, converges on both texts to the same hypotheses, a contradiction. Thus, Claim 3 follows. □

The demonstration of Claim 2 above and of Claim 1 in Theorem 5 immediately imply that feed-back IIMs are not always able to compensate the additional learning power of IIMs that are allowed to store exactly one example.

Corollary 7. $BEM_1 \setminus FB \neq \emptyset$

However, the increase in the learning power obtained by bounded examples memories and feed-back queries is *incomparable* as we shall see. Consequently, there is no unique way to design superior learning algorithms when space limitations are a serious concern. However, our overall goal is a bit more ambitious. We aim to compare the learning power of finite inference from *positive and negative data* (abbr. FIN–INF, cf. Definition 7 below) with those of bounded example memory learning and feed-back identification. As it turns out, feed-back learning from positive data can simulate finite inference from *positive and negative data* while bounded example memory learning cannot. This is interesting, since it addresses the issue whether information presentation can be traded versus memory limitations.

Next, we provide the formal definitions needed. Let $\mathcal{X}$ be a learning domain, and let $\mathcal{C}$ be any concept class defined over $\mathcal{X}$. Furthermore, let $c \in \mathcal{C}$, and let $i = (x_0, b_0), (x_1, b_1), ...$ be an infinite sequence of elements of $\mathcal{X} \times \{+, -\}$ such that $range(i) = \{x_k \mid k \in \mathbb{N}\} = \mathcal{X}$, $i^+ = \{s_k \mid (x_k, b_k) = (x_k, +), k \in \mathbb{N}\} = c$ and $i^- = \{x_k \mid (x_k, b_k) = (x_k, -), k \in \mathbb{N}\} =$ *co-*$c = \mathcal{X} \setminus c$. Then we refer to i as an ***informant***. If c is classified via an informant then we also say that c is represented by ***positive and negative data***. Let c be a concept; by $info(c)$ we denote the set of all informants for c.

Definition 7 (Gold, [7]). *Let $\mathcal{C}$ be an indexable class, let c be a concept, and let $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$ be a hypothesis space.* ***An IIM M FIN–INF–identifies c from informant with respect to $\mathcal{H}$*** *iff for every informant i for c, there exists a $j \in \mathbb{N}$ such that M, when successively fed i outputs the single hypothesis j, $c = h_j$, and stops thereafter.*

Furthermore, M $FIN-INF$–identifies $\mathcal{C}$ with respect to $\mathcal{H}$ iff, for each $c \in \mathcal{C}$, M $FIN-INF$–identifies c with respect to $\mathcal{H}$.

The resulting learning type is denoted by $FIN-INF$.

We start our investigation of whether or not information presentation can be traded versus the mode of convergence and memory limitation with the following observation.

Corollary 8.

(1) $IT \# FIN-INF$

(2) $\bigcup_{k \in \mathbb{N}} BEM_k \# FIN-INF$

Proof. Let $\mathcal{C}_{fin}$ be the class of all finite languages over some fixed alphabet Σ. Obviously, $\mathcal{C}_{fin}$ witnesses $IT \setminus FIN-INF \neq \emptyset$.

For showing $FIN-INF \setminus \bigcup_{k \in \mathbb{N}} BEM_k \neq \emptyset$ we use the indexable class $\mathcal{C}_{CONSV}$. Since $\mathcal{C}_{CONSV} \notin \bigcup_{k \in \mathbb{N}} BEM_k$ (cf. Theorem 5, Claim 2), it remains to show that $\mathcal{C}_{CONSV} \in FIN-INF$. Recall that every language $L \in \mathcal{C}_{CONSV}$ is characterized by its uniquely determined negative example $x \in \{a\}^+ \setminus L$. Clearly, a finite learner has simply to wait until $(x, -)$ appears in the data. Then it outputs the canonical index of the corresponding language $L = \{a\}^+ \setminus \{x\}$ from $\mathcal{C}_{CONSV}$, and stops. □

The latter result is nicely contrasted by our next theorem establishing that feed-back learners capture the whole learning power of finite inference from positive and negative examples.

Theorem 9. $FIN{-}INF \subset FB$.

Proof. By definition $IT \subseteq FB$. Thus, $FB \setminus FIN{-}INF \neq \emptyset$ follows from Corollary 8. Next, we show $FIN{-}INF \subseteq FB$. Let $\mathcal{C} \in FIN{-}INF$, and let M be any IIM finitely learning $\mathcal{C}$ from informant with respect to some hypothesis space $\mathcal{H} = (h_j)_{j\in\mathbb{N}}$. Without loss of generality, we may assume that range$(\mathcal{H}) = \mathcal{C}$ (cf. Lange and Zeugmann [12]). Furthermore, we may also assume M to be total, i.e., for every finite sequence of elements from $\Sigma^+ \times \{+,-\}$ M either outputs a hypothesis or it request the next input (cf. Lange *et al.* [14]).

The desired simulation is based on the following idea. The feed-back learner aims to find an initial segment of an informant for the target language. Such an initial segment can be successively fed to the finitely learning IIM M until M stops or the segment is finished. If M makes an output (referred to as *ordinary hypothesis*), the feed-back learner maintains this guess as long as it is compatible with all the data provided afterwards. However, if a string s is presented that does not belong to the guessed language then the whole process must be iterated. Since a feed-back learner is restricted to one query per one learning stage, we need *auxiliary hypotheses* to memorize the results of the queries made until a new possible initial segment of an informant for the target language is found. Clearly, this idea will only work, if the strategy described above realizes the subset principle.

In order to design the desired feed-back IIM $\hat{M}$ we use the following hypothesis space $\hat{\mathcal{H}} = (\hat{h}_j)_{j\in\mathbb{N}}$. Let $\hat{h}_{2j} = h_j$ for all $j \in \mathbb{N}$, i.e., even indices describe the possible *ordinary hypotheses*. Odd indices are used for *auxiliary hypotheses*. For defining them, let $F_0,\ F_1,\ F_2,\ \ldots$ be any effective enumeration of all finite subsets of Σ^+ including the empty set. For for all $k,\ x \in \mathbb{N}$, set $\hat{h}_{2\langle k,x\rangle+1} = F_k \cup F_x$. The underlying semantics is as follows. The pair (F_k, F_x) represents the fact that that all strings belonging to F_k have already been presented, whereas no string in F_x did appear in the data read so far. For the sake of readability, we use the following convention. When $\hat{M}$ is forced to output an auxiliary hypothesis, say $2\langle k,x\rangle + 1$, we use instead the phrase that $\hat{M}$ is outputting the pair (F_k, F_x). Let $s_0,\ s_1,\ s_2,\ \ldots$ denote any effective repetition free enumeration of all and only the strings in Σ^+. Given any pair (F_k, F_x) that satisfies $F_k \cap F_x = \emptyset$ and $F_k \cup F_x = \{s_j \mid j \leq z = \mathrm{card}(F_k \cup F_x) - 1\}$; we set $i(F_k, F_x)_z = (s_0, b_0),\ \ldots,\ (s_z, b_z)$, where, for all $j \leq z$, $b_j = 1$, if $s_j \in F_k$, and $b_j = 0$ in case that $s_j \in F_x$.

Next, we define the feed-back learner $\hat{M}$. Let $L \in \mathcal{C}$, and let $t = (w_n)_{n\in\mathbb{N}} \in pos(L)$. As usual, we define $\hat{M}$ in stages, where Stage n conceptually describes $\hat{M}_n$.

Stage 0. On input w_0 do the following.

Make the query 's_0.' If the answer is 'NO,' then output the pair $(\emptyset, \{s_0\})$, and goto Stage 1. If the answer is 'YES,' then output the pair $(\{s_0\}, \emptyset)$, and goto Stage 1.

Stage n, $n \geq 1$. $\hat{M}$ receives as input j_{n-1} and the $(n+1)$st element w_n of t.

Case A. j_{n-1} is an ordinary hypothesis.

If $w_n \in \hat{h}_{j_{n-1}}$, set $j_n = j_{n-1}$, output j_n, and goto Stage $n+1$. Otherwise, make the query 's_0.' If the answer is 'NO,' then output the pair $(\emptyset, \{s_0\})$, and goto Stage $n+1$. If the answer is 'YES,' then output the pair $(\{s_0\}, \emptyset)$, and goto Stage $n+1$.

Case B. j_{n-1} is an auxiliary hypothesis, say (P, N).

Test whether or not $w_n \in P \cup N$. If not, then set $P' = P$ and $N' = N$. Otherwise, set $P' = P \cup \{w_n\}$ and $N' = N \setminus \{w_n\}$. Determine $z = \mathrm{card}(P' \cup N')$, and make the query '$s_z$.' If the answer is 'NO,' set $N' = N \cup \{s_z\}$. Else, set $P' = P \cup \{s_z\}$. Determine $i(P', N')_z$, and execute Instruction (β).

(β) Compute successively $M(i(P', N')_0)$, $M(i(P', N')_1)$, ..., $M(i(P', N')_z)$ until M outputs a hypothesis j, say on $i(P', N')_r$, $r \leq z$, or the whole initial segment has been fed. If there was no output or j does not fulfill $P' \subseteq \hat{h}_{2j}$ and $i(P', N')_r^- \cap \hat{h}_{2j} = \emptyset$, output the pair (P', N'), and goto Stage $n+1$.
Otherwise, output the ordinary hypothesis $2j$ and goto Stage $n+1$.

Obviously, $\hat{M}$ is a feed-back IIM. It remains to show that $\hat{M}$ learns L as required. We start with some helpful observations. Suppose $\hat{M}$ outputs in Stage n an auxiliary hypothesis, say (P, N). Let $z = \mathrm{card}(P \cup N)$, and let P' and N' be the corresponding finite sets $\hat{M}$ has generated before executing Instruction (β) within Stage $n+1$. (* Note that $\mathrm{card}(P' \cup N') = z + 1$. *) By construction, in all the Stages $n - z + 1$,..., $n - 1$ the IIM $\hat{M}$ has output auxiliary hypotheses, too. Hence, we have:

Observation 1. $P' \subseteq L$

Observation 2. $P' \cup N' = \{s_j \mid j \leq z\}$

Observation 3. For all $j \leq z$, $s_j \in t_n^+$ implies $s_j \in P'$.

For verifying the latter observation recall that $\hat{M}$ has successively queried $s_0, \ldots, s_z$. Clearly, if s_ℓ, $1 \leq \ell \leq z$, has been presented before the query is made, then the answer is 'YES,' and thus $s_\ell \in P'$. Now assume s_ℓ is queried, say in Stage κ, $\kappa \leq n$, but $s_\ell \notin t_\kappa^+$. Since $s_\ell \in t_{n+1}^+$, the string s_ℓ must appear as input in one of the Stages $\kappa + 1$, ..., $n + 1$. However, then we are in Case B, and hence s_ℓ has to be in P' and cannot belong to N'.

Furthermore, since two successively output auxiliary hypotheses are definitely different, $\hat{M}$ cannot converge to an auxiliary hypothesis.

Claim A. $\hat{M}$ converges.

Let $i(L)$ be the lexicographically ordered informant of L. Let y_0 be the least number y such that M after having successively fed $i(L)_y$ outputs a hypothesis, say j and stops. Since M has to learn L from $i(L)$ such a y must exist, and furthermore, $L = h_j$. Now, let n_0 be the least number n satisfying $i(L)_{y_0}^+ \subseteq t_n^+$.

Suppose $\hat{M}$ has not yet converged when entering Stage n_0. Then there are two possible cases.

Case 1. j_{n_0-1} is an ordinary hypothesis.

Since $\hat{M}$ has not yet converged, this hypothesis must be changed in some subsequent stage, say in Stage $n_0 + \mu$, $\mu \geq 0$. Consequently, $\hat{M}$ begins in Stage $n_0 + \mu$ to query an initial segment of the lexicographically ordered informant for L by asking 's_0,' and it outputs the corresponding auxiliary hypothesis (P, N). Note that the label of s_0 must be correct by the choice of n_0. In Stage $n_0 + \mu + 1$ the next query, i.e., 's_1,' is made. Again, the resulting label of s_1 must be correct. Now, $\hat{M}$ enters Instruction (β). If $y_0 \leq 1$, then M outputs j, and thus, $\hat{M}$ outputs $2j$. Since $L = \hat{h}_{2j}$, this guess is repeated in every subsequent stage. Consequently, $\hat{M}$ converges. On the other hand, if $y_0 > 1$, then, by the choice of y_0, M cannot make an output. Thus, $\hat{M}$ outputs the auxiliary

hypothesis correctly describing the labeling of s_0 and s_1. Iterating this argument, we see that $\hat{M}$ outputs exclusively auxiliary hypotheses until the correct initial segment $i(L)_{y_0}$ is obtained. Then, entering Instruction (β) results in obtaining j. By Observation 1, $\hat{h}_{2j}$ comprises all the positive data seen so far. Finally, $\hat{h}_{2j}$ is disjoint from the data labeled '-,' thus $2j$ passes the test, and $\hat{M}$ converges.

Case 2. j_{n_0-1} is an auxiliary hypothesis.

Now, the same argument applies *mutatis mutandis*. Either $\hat{M}$ produces in some subsequent stage an ordinary hypothesis and converges to it, or we are back to Case 1. This proves Claim A.

Finally, we have to show that the hypothesis $\hat{M}$ converges to is a correct one.

Claim B. If $\hat{M}$ converges, say to $2k$, then $L = \hat{h}_{2k}$.

Since $\hat{M}$ converges, we know that $2k$ is an ordinary hypothesis. Suppose $L \neq \hat{h}_k$. Let m be the first stage in which k is output. By construction, the hypothesis output in Stage $m-1$ must have been an auxiliary hypothesis, say (P, N). Let $z = \mathrm{card}(P \cup N)$, and let P' and N' be the corresponding finite sets $\hat{M}$ has generated before executing Instruction (β) within Stage m. By $i(P', N')_z$ we denote the corresponding initial segment of some informant. Furthermore, let $r \leq z$ be such that $k = M(i(P', N')_r)$. If $i(P', N')_r$ is an initial segment of $i(L)_{y_0}$, then $r = y_0$, and we are done.

Now assume $i(P', N')_r$ to be not an initial segment of $i(L)_{y_0}$. Then $i(P', N')^+_r \not\subseteq i(L)^+_{y_0}$ or $i(P', N')^-_r \not\subseteq i(L)^-_{y_0}$. Since $\hat{M}$ has verified $P' \subseteq \hat{h}_{2k}$ and $i(P', N')^-_r \cap \hat{h}_{2k} = \emptyset$, we know that $L \neq \hat{h}_{2k}$. By Observation 1, we additionally have $P' \subseteq L$, thus we may conclude $i(P', N')^-_r \cap L \neq \emptyset$. Because of $i(P', N')^-_r \cap \hat{h}_{2k} = \emptyset$, we know that there exists a string $s \in (L \cap i(P', N')^-_r) \setminus \hat{h}_{2k}$. Taking into account that $2k$ has been first produced in Stage m, we are done if $s \in t^+_{m+\rho} \setminus t^+_m$ for some $\rho \geq 1$. Finally, assume $s \in t^+_m$. By Observation 3, the uniquely determined index μ with $s = s_\mu$ must satisfy $\mu > z$, since otherwise $s \in P'$, and $P' \subseteq \hat{h}_{2k}$ would be contradicted. However, $s \in (L \cap i(P', N')^-_r)$ yielding $\mu \leq z$ by Observation 2. This contradiction proves Claim B. □

The issue whether information presentation can be traded versus the mode of convergence has been treated in Lange *et al.* [14], too. In particular, in [14] we addressed the question to what extend the subset principle must be weakened if only positive data are available. The results obtained in this context imply $FIN\text{-}INF \subset CONSV$. Because of $FB \subset CONSV$ (cf. Theorem 5, Assertion (2)), our last theorem strengthen this result.

Finally, putting the results obtained together, we obtain the already announced result that the increase in the learning power obtained by bounded example memories and feedback queries is incomparable.

Theorem 10.

(1) $FB \setminus \bigcup_{k \in \mathbb{N}} BEM_k \neq \emptyset$,

(2) $BEM_{k+1} \# FB$ *for all* $k \in \mathbb{N}$.

Proof. Clearly, Assertion (1) follows directly from Theorem 9 and Corollary 8. Furthermore, $BEM_1 \setminus FB \neq \emptyset$ by Corollary 7, and hence, Assertion (2) follows. □

3.3. A Sufficient Condition for Incremental Learning

There are several well-known criteria that ensure learnability in the limit of indexable classes from positive data, i.e., finite thickness and finite elasticity. Both conditions are sufficient but not necessary. Hence, it is only natural to ask whether or not these conditions guarantee any form of incremental learning, too. Unfortunately, the general answer is negative. However, a natural sharpening of these conditions directly yields sufficient conditions for all models of incremental learning introduced above.

Definition 8 (Angluin [2]). *Let $\mathcal{C}$ be an indexable class. $\mathcal{C}$ has **finite thickness** if and only if for every $x \in \mathcal{X}$ there are at most finitely many $c \in \mathcal{C}$ satisfying $x \in c$.*

Proposition 2. *There is an indexable class $\mathcal{C} \notin CONSV$ which has finite thickness.*

Proof. Consider the following indexable class $\mathcal{C}$ of languages $L_{\langle k,j \rangle}$ defined as follows. For all $k \in \mathbb{N}$, we set $L_{\langle k,0 \rangle} = \{a^k b^n \mid n \in \mathbb{N}\}$. Note that $a^0 = \varepsilon$ by convention. For all $k \in \mathbb{N}$ and all $j > 0$, we distinguish the following cases:

Case 1. $\neg\, \Phi_k(k) \leq j$

Then we set $L_{\langle k,j \rangle} = L_{\langle k,0 \rangle}$.

Case 2. $\Phi_k(k) \leq j$

We distinguish the following subcases.

Subcase 2.1. $j < 2\Phi_k(k)$

Let $r = 2\Phi_k(k) - j$. We set $L_{\langle k,j \rangle} = \{a^k b^m \mid 1 \leq m \leq r\}$.

Subcase 2.2. $j \geq 2\Phi_k(k)$

Then we set $L_{\langle k,j \rangle} = \{a^k b\}$.

Finally, let $\mathcal{L} = (L_{\langle k,j \rangle})_{j,k \in \mathbb{N}}$, and let $\mathcal{C} = \text{range}(\mathcal{L})$. Moreover, $\mathcal{C}$ has finite thickness by construction. On the other hand, we know that $\mathcal{C} \notin CONSV$ (cf. [12], Theorem 1). □

Because of Theorem 5 we may conclude:

Corollary 11. *There is an indexable class $\mathcal{C}$ having finite thickness which does not belong to $IT \cup FB \cup \bigcup_{k \in \mathbb{N}} BEM_k$.*

Next, we define recursive finite thickness. Let $\mathcal{X}$ be any recursively enumerable learning domain, and let $x_0,\ x_1,\ x_2,\ \ldots$ be any effective enumeration of all elements in $\mathcal{X}$. Furthermore, assume an effective enumeration $N_0,\ N_1,\ N_2,\ \ldots$ of all finite subsets of $\mathbb{N}$.

Definition 9. *Let $\mathcal{C}$ be an indexable class. $\mathcal{C}$ has **recursive finite thickness** provided there are an indexing $c_0,\ c_1,\ c_2,\ \ldots$ of $\mathcal{C}$ and a total recursive function g such that, for all $m,\ k \in \mathbb{N}$, $x_m \in c_k$ if and only if $k \in N_{g(m)}$ or there is a $j \in N_{g(m)}$ with $c_j = c_k$.*

It is easy to verify that the class of all concepts describable by a monomial, a k-CNF, a k-DNF, a k-decision list, respectively have recursive finite thickness. The pattern languages provide another interesting example of a concept class having recursive finite thickness. The following theorem establishes the iterative learnability of all these concept classes.

Theorem 12. *Let $\mathcal{C}$ be an indexable class. If $\mathcal{C}$ has recursive finite thickness, then $\mathcal{C} \in IT$.*

Proof. Let $c_0,\ c_1,\ c_2,\ \ldots$ be an indexing of $\mathcal{C}$ and let g be the corresponding recursive

function which satisfies the requirements of Definition 9. For showing $\mathcal{C} \in IT$ we choose the following hypothesis space $\mathcal{H} = (h_j)_{j \in \mathbb{N}}$. For every $j \in \mathbb{N}$, let $h_j = \bigcap_{k \in N_{g(j)}} c_k$. We define an IIM M IT–inferring $\mathcal{C}$ with respect to $\mathcal{H}$. Let $c \in \mathcal{C}$ and let $t = (s_j)_{j \in \mathbb{N}} \in pos(c)$. The IIM M is defined in stages, where Stage n conceptually describes M_n.

Stage 0. M receives as input s_0.
Determine the unique $m \in \mathbb{N}$ with $x_m = s_0$. Output $g(m)$, and goto Stage 1.

Stage n, $n \geq 1$. M receives as input its last guess, say j, and the $(n+1)$st element s_n of t.

Case 1. $s_n \in h_j$
Output j, and goto Stage $n+1$.

Case 2. $s_n \notin h_j$
For all $k \in N_j$, test whether or not $s_n \in c_k$. Collect all k successfully passing the test within the set $CONS$. Determine the uniquely defined index z for $CONS$, i.e., $N_z = CONS$. Output z, and goto Stage $n+1$.

Now it is straightforward to show that M iteratively infers $\mathcal{C}$ with respect to $\mathcal{H}$. We omit the details. □

The proof given above has some interesting features we want to point to. First of all, the learning algorithm produces its hypotheses in a rather *constructive* manner. This nicely contrasts the enumerative character of many inference procedures often provided in abstract studies within Gold's [7] model (cf., e.g., [4, 7, 16]). In contrast, our general learning algorithm immediately produces a finite subspace of hypotheses from which it computes its actual guess. Subsequently, it *deletes* all nonrelevant hypotheses from this subspace. Moreover, the algorithm learns by *generalization*, i.e., the sequence of its guesses constitutes an augmenting chain of concepts. As a matter of fact, the converse is also true. Whenever the learning process can be exclusively performed by generalization, then one can learn iteratively, too (cf. [11]). However, the generality of the result above does not always yield the most effective iterative learning algorithm. For example, a straightforward application of Valiant's [20] proof technique directly yields iterative learning algorithms for the class of all concepts describable by a k-CNF and k-DNF, respectively, that are much more efficient. Another example are the pattern languages. In this case, Lange and Wiehagen's [10] iterative learning algorithm is the much better choice (cf. Zeugmann [26] for a detailed analysis).

As our next result states, recursive finite thickness is only a sufficient criterion that ensures the learnability by iterative IIMs.

Theorem 13. *There is an indexable class $\mathcal{C} \in IT$ which does not have recursive finite thickness.*

Proof. For all $j \in \mathbb{N}$, let $L_0 = \{a\}^+$, and $L_{j+1} = \{a^m \mid 1 \leq m \leq j+1\} \cup \{b^{j+1}\}$. Let $\mathcal{C}$ denote the collection of all those languages L_j. Obviously, there are infinitely many different languages which contain the string a. Thus, $\mathcal{C}$ even does not have finite thickness. On the other hand, $\mathcal{C} \in IT$ (cf. Lange and Zeugmann [11]). □

Next, we consider finite elasticity introduced by Wright [25].

Definition 10. *Let $\mathcal{C}$ be an indexable family. $\mathcal{C}$ has* ***infinite elasticity*** *if and only if there are an infinite sequence of strings x_0, x_1, x_2, ... and an infinite sequence of concepts c_0, c_1, c_2, ... each in $\mathcal{C}$ such that, for all $n \in \mathbb{N}^+$, $\{x_0, \ldots, x_{n-1}\} \subseteq c_n$ but $x_n \notin c_n$. $\mathcal{C}$ has* ***finite elasticity*** *provided that $\mathcal{C}$ does not have infinite elasticity.*

Obviously, finite thickness implies finite elasticity. Therefore, Corollary 11 yields:

Corollary 14. *There is an indexable class $\mathcal{C} \notin IT \cup FB \cup \bigcup_{k \in \mathbb{N}} BEM_k$ which has finite elasticity.*

On the other hand, the indexable class $\mathcal{C}$ used in the demonstration of Theorem 13 does not have finite elasticity as well. For seeing this, set $x_j = a^{j+1}$ and $c_j = L_j$ for all $j \in \mathbb{N}$. By construction, $\{x_0, \ldots, x_{n-1}\} \subseteq c_n$ but $x_n \notin c_n$. Thus $\mathcal{C}$ has infinite elasticity. Consequently:

Corollary 15. *There is an indexable class $\mathcal{C} \in IT$ which does not have finite elasticity.*

4. Conclusions and Open Problems

During the last decade algorithmic learning has attracted a continuously growing interest in the computer science community. Additionally, machine learning techniques are sought after in a wider range of industrial and scientific applications, e.g., in knowledge engineering, in robotics, in pattern recognition, in financial prediction, in molecular biology, in natural language processing, and in machine discovery. Since every practical learning system has to deal with the limitations of space available, incremental learning techniques are of special interest. Moreover, it is well-known that too little information causes learning systems to fail. On the other hand, too much information may also lead to a degrading performance, a loss of efficiency, and it may even affect the accuracy. Therefore, it is of central importance to gain a better understanding of what data must be preserved during the learning process, and of what information can be overlooked. Clearly, these problems have various facets, and several of them have been studied in inductive inference (cf., e.g., Wiehagen and Zeugmann [24] and the references therein).

The present paper addresses some of these problems from a new perspective by providing a systematic study of incremental learning for indexable concept classes. Different models of incremental learning from positive data have been defined and investigated. These model differ in the way and extent they restrict the accessibility of the input data stream. We distinguished between iterative learning, bounded example memory inference and feed-back identification.

An iterative learner is required to produce its actual hypothesis exclusively from its previous guess and the next example presented. Bounded example memory and feed-back learning generalize iterative inference by allowing to store an *a priori* bounded number of carefully chosen examples and asking whether or not a particular element did already appear in the input data provided so far, respectively.

As it turned out, all the formal models defined correspond to learning scenarios that are generally less powerful than conservative learning (cf. Theorem 5). On the other hand, by realizing a suitable interplay between the learning algorithm and the hypothesis space chosen, incremental learning may outperform conservative learning, too (cf. Theorem 1). Moreover, as the proof of Theorem 1 shows *redundancy* in the hypothesis space may

seriously increase the learning capabilities of incremental learners. Future research should address the problem of what properties hypothesis spaces must have to be well suited for incremental learning.

Moreover, both feed-back learning and bounded example memory inference are more powerful than iterative learning. In particular, we established a new infinite hierarchy of more and more powerful bounded example memory learners parametrized by the number of examples storable. However, feed-back learning and bounded example memory inference extend the learning capabilities of iterative learners in different directions. This insight allows the conclusion that there is no unique way to design superior incremental learning algorithms.

Finally, an easy verifiable sufficient condition for incremental learning has been elaborated. Applying this criterion, the iterative learnability of all concepts describable by a monomial, a k-CNF, a k-DNF, a k-decision list, respectively, and of all the pattern languages can be shown. The importance of this criterion is mainly based on its simplicity. Once it is known that there exist an iterative learning algorithm future research can concentrate on improving its efficiency. Clearly, it would be highly desirable to have similar conditions for feed-back learning and bounded example memory inference. Ideally, one should elaborate conditions that are both necessary and sufficient for the different models of incremental learning.

5. References

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