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ON THE NONBOUNDABILITY OF TOTAL EFFECTIVE OPERATORS

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Introduction

In this paper we attempt to analyze, in some more detail, the difference between total effective operators (defined on all general recursive functions) and general recursive operators (computable operators defined on all total functions). It is well-known that not every total effective operator is general recursive; the partial recursive operator obtained by the KREISEL-LACOMBE-SHOENFIELD theorem [2] does not always map every total function to a total function. Therefore we are particularly interested in learning, in which sense total effective and general recursive operators are extremely different. In this content it has been known, that every general recursive operator is bounded (cf. Definition 4) by a general recursive monotone operator; this result was proved by MEYER and FISCHER [3]. HELM [1] showed that an analogous theorem for total effective operators cannot be obtained. Till now it has been unknown whether this result can be stengthened or not. In response to a related question stated by HELM, we will show that in general total effective operators are not boundable even not in a very weak sense (cf. Theorem 1 and 2). Our theorems given below will show that in proving theorems concerning computable operators, it does neither suffice to consider only general recursive operators; nor to consider only monotone or quasi-monotone operators.

We assume the reader is familar with ROGERS [4]. Now we shall give some basic notations and definitions which will be used in this paper.

1. Basic Notations and Definitions

$\mathsf{N} = \{0, 1, 2, \ldots\}$ denotes the set of all natural numbers. P and R mark the class of all partial recursive and general recursive functions, respectively of one variable over N. The class of all general recursive predicates over N is denoted by R_0 (i.e. $f \in \mathsf{R}_0$ iff $f \in \mathsf{R}$ and $\mathrm{rg}(f) \subseteq \{0, 1\}$). An acceptable Gödel numbering of P is denoted by φ. The i-th partial recursive function is then marked by φ_i. The abbreviation a.e. stands for "almost everywhere" and means for all but finitely many values. We write i.o. as an abbreviation for "infinitely often". If $\mathfrak{O}$ is an operator which maps functions to functions, we write $\mathfrak{O}(f, x)$ to denote the value of the function $\mathfrak{O}(f)$ at point x.

Definition 1. A mapping $\mathfrak{O}\colon \mathsf{P} \to \mathsf{P}$ is an *effective operator* iff there exists an $g \in \mathsf{P}$ such that if $\varphi_i \in \mathrm{dom}(\mathfrak{O})$, $g(i)$ is defined and $\mathfrak{O}(\varphi_i) = \varphi_{g(i)}$. The operator $\mathfrak{O}$ is *total effective* provided that $R \subseteq \mathrm{dom}(\mathfrak{O})$ and $\varphi_i \in \mathsf{R}$ implies $\mathfrak{O}(\varphi_i) \in \mathsf{R}$.

Definition 2. An operator $\mathfrak{O}\colon \mathsf{P} \to \mathsf{P}$ is *monotone* if for all functions $f, g \in \mathrm{dom}(\mathfrak{O})$: if $f(x) \leqq g(x)$ a.e., then $\mathfrak{O}(f, x) \leqq \mathfrak{O}(g, x)$ a.e.

Definition 3. An operator $\mathfrak{O}\colon \mathsf{P} \to \mathsf{P}$ is *quasimonotone* if for all functions $f, g \in \mathrm{dom}(\mathfrak{O})$: if $f(x) \leqq g(x)$ a.e., then $\mathfrak{O}(f, x) \leqq \mathfrak{O}(g, x)$ i.o.

Definition 4. An operator $\mathfrak{O}$ is said to be *(weakly) boundable* if there exists an operator $\mathfrak{O}'$ such that

(1) $\mathrm{dom}(\mathfrak{O}) \subseteqq \mathrm{dom}(\mathfrak{O}')$,

(2) For every function $f \in \mathrm{dom}(\mathfrak{O})$ it holds that $\mathfrak{O}(f, x) \leqq \mathfrak{O}'(f, x)$ a.e. (i.o.).

2. Results

Now we will research into the following two problems:

(1) Can every total effective operator be weakly bounded by a total effective monotone operator?

(2) Is it possible to bound every total effective operator by a total effective quasimonotone operator?

In response to these two questions, we construct a total effective operator which neither can be weakly bounded by any total effective monotone operator nor can be bounded by a total effective quasimonotone operator.

At first we shall show that the first problem can be reduced to the second one.

Theorem 1. *Let $\mathfrak{O}$ be any total effective operator. Then the following holds: If the operator $\mathfrak{O}$ is weakly boundable by a total effective monotone operator $\mathfrak{G}$, then $\mathfrak{O}$ can be bounded by a quasimonotone total effective operator $\mathfrak{Q}$.*

Proof. Let $\mathfrak{O}$ be any arbitrarily fixed total effective operator. Suppose that the total effective monotone operator $\mathfrak{G}$ weakly bounds the operator $\mathfrak{O}$; that means $\mathrm{dom}(\mathfrak{O}) \subseteqq \mathrm{dom}(\mathfrak{G})$ and for every function $f \in \mathrm{dom}(\mathfrak{O})$ it holds that

(A) $\qquad \mathfrak{O}(f, x) \leqq \mathfrak{G}(f, x)$ i.o.

We define now an operator $\mathfrak{Q}$ as follows: $\mathfrak{Q}(f, x) = \max\{\mathfrak{O}(f, x), \mathfrak{G}(f, x)\}$ for all functions $f \in \mathsf{P}$ and all $x \in \mathsf{N}$. Then by construction $\mathfrak{Q}$ is clearly total effective and satisfies $\mathfrak{O}(f, x) \leqq \mathfrak{Q}(f, x)$ for all functions $f \in \mathrm{dom}(\mathfrak{O})$ and all $x \in \mathsf{N}$. Therefore, the operator $\mathfrak{Q}$ bounds the operator $\mathfrak{O}$. It remains to show that $\mathfrak{Q}$ is a quasimonotone operator.

Let f, g be two arbitrarily fixed functions from $\mathrm{dom}(\mathfrak{Q})$ with $f(x) \leqq g(x)$ a.e. Since $\mathfrak{G}$ is a monotone operator it holds

(B) $\qquad \mathfrak{G}(f, x) \leqq \mathfrak{G}(g, x)$ a.e.

By (A) and by construction of the operator $\mathfrak{Q}$ we get $\mathfrak{Q}(f, x) = \mathfrak{G}(f, x)$ i.o. and $\mathfrak{Q}(f, x) \geqq \mathfrak{G}(f, x)$ for all x and every function $f \in \mathrm{dom}(\mathfrak{Q})$. Thus we obtain by using (B) $\mathfrak{Q}(f, x) = \mathfrak{G}(f, x) \leqq \mathfrak{G}(g, x) \leqq \mathfrak{Q}(g, x)$ i.o. and therefore $\mathfrak{Q}$ is quasimonotone. □

Now we give an example of a total effective operator which cannot be bounded by any quasimonotone operator.

Theorem 2. *There exists a total effective operator $\mathfrak{O}$ which is no boundable by any total effective quasimonotone operator $\mathfrak{Q}$.*

Proof. Let φ be a fixed acceptable Gödel numbering of P and set $K = \{k \mid \varphi_k(k)$ converges$\}$. It is well-known that K can be effectively enumerated. Let $k_0, k_1, k_2, \ldots$ be a fixed effective enumeration of K. Define

$$\mathfrak{O}(f, x) = \text{the least } j \text{ such that } k_j > x \text{ and } f(k_j) = \varphi_{k_j}(k_j).$$

By the Theorem of Rice, $\{i \mid \varphi_i = f\}$ is infinite for any function $f \in \mathsf{P}$ and thus operator $\mathfrak{O}$ is clearly total effective. To see that it satisfies the other requirements, we prove the following

Lemma. *Let $\mathfrak{O}$ be the above defined operator. Then for every function $h \in \mathsf{R}$ there is a predicate $f \in \mathsf{R}_0$ such that for all x, $\mathfrak{O}(f, x) > h(x)$.*

Proof. Let h be an arbitrarily fixed function from R. Define the function h' as follows: $h'(0) = h(0)$ and $h'(x + 1) = h'(x) + h(x + 1)$. By construction, $h' \in \mathsf{R}$ and $h(x) \leqq h'(x)$ for all x. Furthermore h' is monotone. To define the wanted predicate f let $K^x = \{k_0, \ldots, k_x\}$ and, for all x, let

$$f(x) = \begin{cases} 0, & \text{if } x \notin K^{h'(x)} \text{ or } x \in K^{h'(x)} \text{ and } \varphi_x(x) \neq 0 \\ 1, & \text{if } x \in K^{h'(x)} \text{ and } \varphi_x(x) = 0. \end{cases}$$

Clearly $\mathrm{rg}(f) \subseteqq \{0, 1\}$.

Claim 1. *The predicate f is general recursive.*

Let x be a fixed element of N. It is to show that $f(x)$ is defined and computable. At first compute $h'(x)$. This is effectively possible since $h' \in \mathsf{R}$. Thus $K^{h'(x)}$ is effectively computable and finite. Therefore, it is decidable whether $x \in K^{h'(x)}$. In the case that $x \notin K^{h'(x)}$, the predicate f is already defined at the point x. Otherwise, by the definition of K it holds that $\varphi_x(x)$ converges. Thus it is decidable whether $\varphi_x(x) = 0$ or not. So we have $f \in \mathsf{R}$.

Claim 2. *$\mathfrak{O}(f, x) > h'(x)$ for all x.*

Since $\mathfrak{O}$ is a total effective operator and $f \in \mathsf{R}_0$, we have $\mathfrak{O}(f) \in \mathsf{R}$. Thus $\mathfrak{O}(f, x)$ is defined for all x. Let $\mathfrak{O}(f, x) = j$. It remains to show that $j > h'(x)$. Suppose the converse, i.e. $j \leqq h'(x)$. By the definition of the operator $\mathfrak{O}$ it follows that $k_j > x$ and therefore (since h' is monotone) $j \leqq h'(k_j)$. Thus, $k_j \in K^{h'(k_j)}$. By construction of the predicate f we obtain $\varphi_{k_j}(k_j) = 0$ iff $f(k_j) \neq 0$, this is a contradiction. So we get $h'(x) < j = \mathfrak{O}(f, x)$. To finish the proof of the lemma we use the fact that $h'(x) \geqq h(x)$ for all x. Thus $\mathfrak{O}(f, x) > h(x)$ for all x. To obtain the desired statement of our theorem it suffices to fix a function $r \in \mathsf{R}$ with $r(x) \geqq 1$ for all x. We set $h(x) = \mathfrak{Q}(r, x)$, where $\mathfrak{Q}$ is an arbitrarily fixed total effective quasimonotone operator. Thus it holds that the function h is general recursive, and for any predicate $f \in \mathsf{R}_0$ we have $\mathfrak{Q}(f, x) \leqq \mathfrak{Q}(r, x)$ i.o. Therefore, by our lemma, we obtain that there is a predicate $f \in \mathsf{R}_0$ with $\mathfrak{O}(f, x) > \mathfrak{Q}(r, x) \geqq \mathfrak{Q}(f, x)$ i.o. This completes the proof. □

Corollary. *Let $\mathfrak{O}$ be the above defined operator. Then it holds: The operator $\mathfrak{O}$ is not weakly boundable by any total effective monotone operator.*

Proof. It is an immediate consequence of Theorem 1 and 2.

Finally we want to discuss these results. First, we remark that obviously there are infinitely many predicates such that for the above defined operator $\mathfrak{O}$ and any total effective quasimonotone operator $\mathfrak{Q}$ it holds that $\mathfrak{O}(f, x) > \mathfrak{Q}(f, x)$ i.o. Next, it is easy to show that any total effective quasimonotone operator $\mathfrak{Q}$ does not only fail to bound the operator $\mathfrak{D}$ on R_0, it fails to do so on arbitrarily large functions. Then, we want to point out that Theorem 2 is even true if we replace the phrase "any total effective quasimonotone operator $\mathfrak{Q}$" by "any total effective quasimonotone operator $\mathfrak{Q}$ relative to R_0". (An operator $\mathfrak{Q}$ is called *quasimonotone relative to* R_0 if there is a function $r \in \mathsf{R}$ such that $r(x) \geqq 1$ for all x, and for any predicate $f \in \mathsf{R}_0$ it holds that $\mathfrak{Q}(f, x) \leqq \mathfrak{Q}(r, x)$ i.o.) The proof is, in essential, the same.

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