Classifying Predicates and Languages*

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Abstract. The present paper studies a particular collection of classification problems, i.e., the classification of recursive predicates and languages, for arriving at a deeper understanding of what classification really is. In particular, the classification of predicates and languages is compared with the classification of arbitrary recursive functions and with their learnability. The investigation undertaken is refined by introducing classification within a resource bound resulting in a new hierarchy. Furthermore, a formalization of *multi–classification* is presented and completely characterized in terms of standard classification. Additionally, *consistent* classification is introduced and compared with both resource bounded classification and standard classification. Finally, the classification of families of languages that have attracted attention in learning theory is studied, too.

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1. Introduction

Learning and *classification* have attracted considerable attention by computer scientists, both in theory and practice. *Inductive inference* is an important aspect of learning that has been widely studied (cf. Angluin and Smith (1983, 1987)). The inductive inference problem is to take finite samples of some target concept and to *generalize* an algorithm that can produce *all* samples of the same concept. Hence, inductive inference may be regarded as the most general framework to study the *generalization problem* (cf. Michalski et al. (1983)). The *classification problem* may be described as follows: Given a number of, usually finite, choices, one takes finite samples of a target concept and has to find out algorithmically to which of the possible choices the concept belongs (cf. Duda and Hart (1973); Ben-David (1992) addresses the problem of infinitely many classes with no computational requirements imposed on the classifying algorithms).

Recently, the problem of classification has been compared with the inductive inference problem in a recursion theoretic setting (cf. Wiehagen and Smith (1995)). In that paper a new formalization of classification was introduced. The algorithms studied there classified arbitrary functions. Herein, we consider the classification of $\{0,1\}$ valued functions. These functions are often called *predicates* as they represent binary decisions on the input. By utilizing the isomorphism between strings of symbols and the natural numbers, the predicates over the natural numbers also represent *formal languages*. Hence, we simultaneously study the classification of predicates and languages. Previous studies of learning have revealed some subtle and not so subtle differences between the learning of functions versus the learning of languages. We are interested in learning the differences and similarities between the classification of recursive predicates and arbitrary recursive functions.

Our interest in such a restricted class of functions stems from consideration of some practical classification problems where the input is similarly restricted. Many such classification problems exist. We give two examples. Consider the problem of medical diagnosis. Every patient can be abstractly modeled as a mapping from symptoms to an indication of their presence or absence. The goal is to classify the collection of symptoms as to which malady is manifesting them. While there are clearly only finitely many possible symptoms, an *a priori* restriction on their number would place a technological bound on the applicability of our results. Our orientation is theoretical and we are hoping for results with longevity in the midst of a rapidly changing technological base.

As another example of when classification input can be restricted to examples of predicates consider the problem of battlefield arena classification. The presence or absence of various types of weaponry and terrain features may be sufficient to produce a useful classification. In this example, we formally consider the classification of functions that map the myriad of weapons and landscape parameters to single bits, representing their inclusion in the scene.

In both of the above examples, it appears that they would still be within the scope of classification results that applied only to predicates. The hope is that by pushing the study of classification of predicates far enough, a difference with the unrestricted classification will be revealed that suggests an enhancement that can be made in proposed solutions to problems such as the ones described above.

Aside from having their input restricted to predicates, there is another feature shared by the two examples above. For a given set of characteristics, there may be several simultaneously appropriate classifications. In the case of medical diagnosis, it is crucial to know *all* of the potential diseases that may be manifesting the observed symptoms. We consider this variant of the classification problem called *multi*-classification where the object is to identify *all* the appropriate classifications. The multi-classification problem is known to be very difficult (cf. Bylander et al. (1991), Cho and Reggia (1993), and Freivalds and Hoffmann (1994)). We explain this difficulty by proving that every multi-classification problem is equivalent to a classification problem where the cardinality of the potential choices increases exponentially.

The technical approach of the present paper has five parts. Firstly, we apply the previously developed formalism to investigate the classification of $\{0,1\}$ valued recursive functions. Secondly, we compare the power of classification algorithms that are allowed to produce only a single guess (finite classification, cf. Definition 2, the $c = 0$ case) with those that may change their mind a predetermined fixed number of times (cf. Definition 2, the $c \in \mathbb{N}$ case). Thirdly, we consider the case were the classification algorithm is to produce *all* of the correct classifications (cf. Definition 3). Fourthly, we introduce the notion of *consistent* classification (cf. Definition 4) and compare it with general classification. The goal of these studies is to find some other reasonable restrictions that may make a profound difference on the classifiability of certain sets of functions. Fifthly, we study classification of families of languages that have received considerable attention in learning theory and elsewhere in computer science.

2. Technical Preliminaries

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of all natural numbers. Members of $\mathbb{N}$ will serve as names for programs. The function computed by program i will be denoted by φ_i. Most reasonable ways of assigning names to programs result in a list φ_0, φ_1, ... called an *acceptable programming system* (cf. Machtey and Young (1978) and Smith (1994)). By $\mathcal{R}, \mathcal{R}^2$ we denote the class of all recursive functions of one, two arguments, respectively. The class of $\{0,1\}$ valued recursive functions, our model of both predicates and languages, is denoted by $\mathcal{R}_{0,1}$. For $S \subseteq \mathcal{R}$, let $\rho(S) = S \cap \mathcal{R}_{0,1}$ denote the restriction of S to predicates. A set is *recursively enumerable* (r.e.) iff it is the domain of some φ_i. A more intuitive, equivalent characterization of the r.e. sets is to call a set r.e. if it is either the range of a recursive function or it is empty. In this way it is easy to see that the r.e. sets are the ones for which we can

write a procedure that prints out, eventually, all and only members of the set in question. The i^{th} r.e. set, corresponding to the domain of φ_i, is denoted by W_i. Let $K = \{i \mid i \in \mathbb{N},\ \varphi_i(i) \text{ is defined}\}$ denote the halting set. Clearly, K is recursively enumerable but not decidable (cf. Machtey and Young (1978) and Smith (1994)).

A collection C of recursive functions is an *r.e. set of recursive functions* iff there is a recursive function f such that $C = \{\varphi_{f(i)} \mid i \in \mathbb{N}\}$. By *NUM* we denote the collection of all r.e. sets of recursive functions. Another equivalent definition of *NUM* which is more convenient for most of our purposes is the following. Let $\psi \in \mathcal{R}^2$. Then we call ψ a numbering. Let $\psi_i = \lambda x \psi(i, x)$ denote the function from $\mathcal{R}$ which is computed by program i. The basic difference between the effective lists $\psi_0, \psi_1, \ldots$ and $\varphi_0, \varphi_1, \ldots$ above consists in that φ enumerates all *partial recursive* functions whereas ψ enumerates some set of exclusively *recursive* functions. Let $\mathcal{R}_\psi = \{\psi_i \mid i \in \mathbb{N}\}$ denote the class of the recursive functions enumerated by the numbering ψ. Then $NUM = \{\mathcal{R}_\psi \mid \psi \in \mathcal{R}^2\}$ is a definition of *NUM* equivalent to that above.

For a function $f \in \mathcal{R}$ and $n \in \mathbb{N}$, let $f^n = \text{cod}(f(0), f(1), \ldots, f(n))$, where cod denotes a computable and bijective mapping from the set $\mathbb{N}^*$ of all finite sequences of natural numbers onto $\mathbb{N}$. Analogously, $\{0,1\}^*$ denotes the set of all finite $\{0,1\}$ valued sequences.

Whenever appropriate we shall represent a recursive function by the *sequence of its values*. For example, the sequence $0^2 1^3 2^\infty$ denotes the function

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 2, \\ 1, & \text{if} \quad 2 \leq x < 5, \\ 2, & \text{otherwise.} \end{cases}$$

Let $f, g \in \mathcal{R},\ n \in \mathbb{N}$. Then $f =_n g$ iff $f(x) = g(x)$ for any $x \leq n$; $f \neq_n g$ iff not $f =_n g$. Let $U \subseteq \mathcal{R}$ and $h \in \mathcal{R}$. Then h is called an accumulation point of U iff for any $n \in \mathbb{N}$, there is $f \in U$ such that $f =_n h$ but $f \neq h$. Note that we do not require h to be an element of U. Let $Acc(U)$ denote the set of all accumulation points of U. Subset is denoted by $\subseteq$ and $\subset$ denotes *proper* subset. "Almost all" is interpreted as "all but finitely many."

Next, we formalize the models of identification and classification mentioned in the introduction. As in Gold (1967) we define an *inductive inference machine* (abbr. IIM) to be an algorithmic device which works as follows: The IIM takes as its input larger and larger initial segments of the graph of a function and it either requests the next function value, or it first outputs a hypothesis, i.e., a name of a program, and then it requests the next function value.

A classification machine (abbr. CM) takes as input the graph of a function (as IIMs do) and it either requests the next function value, or it first outputs an integer chosen from a finite set, and then it requests the next function value.

Let M be an IIM or a CM. Furthermore, let i and j be two consecutive hypotheses

produced by M. We say that M changes its mind, or synonymously, M performs a mind change, iff $i \neq j$. When dealing with mind changes, it is technically much more convenient to require the IIMs to behave as follows. Let f be a recursive function. If $\hat{M}$ on $f(0), \ldots, f(n)$ outputs its first guess, then it has to output a hypothesis at any subsequent step. It is easy to see that any IIM M may be straightforwardly converted into an IIM $\hat{M}$ behaving as required such that both machines produce the same sequence of mind changes.

We start with the formalization of learning. The following definition is due to Gold (1967), Barzdin (1971) and Blum and Blum (1975).

Definition 1. *Let $U \subseteq \mathcal{R}$ and let $c \in \mathbb{N} \cup \{*\}$. The class U is said to be learnable (or inferible) with at most c mind changes iff there is an IIM M such that for all $f \in U$,*

(1) *there is a j such that $\varphi_j = f$ and $M(f^n) = j$ for almost all $n \in \mathbb{N}$,*

(2) *M, when successively fed $f(0)$, $f(1)$, ... performs at most c ($c = *$ means at most finitely many) mind changes, i.e.,* $\mathrm{card}(\{n \mid M(f^n) \neq M(f^{n+1})\}) \leq c$.

If U can be learned by an IIM M with at most c mind changes, then we write $U \in EX_c(M)$. The class of all sets of recursive functions that are learnable with at most c mind changes is denoted by EX_c, an abbreviation for *explains* as a program for f can be regarded as an explanation of the set of examples constituting the graph of f (cf. Case and Smith (1983)). If U can be learned with 0 mind changes, then we also say that U is *finitely* learnable. Moreover, we set $FIN = EX_0$. If $c = *$, then we usually omit the lower index and simply say that U can be learned or $U \in EX$.

Next, we formalize classification of finitely many sets.

Definition 2. *Let $k \geq 2$, $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$ and $S = S_0 \cup \ldots \cup S_{k-1}$. Furthermore, let $c \in \mathbb{N} \cup \{*\}$. Then $(S_0, \ldots, S_{k-1})$ is said to be classifiable with at most c mind changes iff there is a CM M such that for all $f \in S$,*

(1) *for all $n \in \mathbb{N}$, whenever M, on input f^n, outputs a hypothesis j, then $j \in \{0, \ldots, k-1\}$,*

(2) *there is a j such that $f \in S_j$ and $M(f^n) = j$ for almost all $n \in \mathbb{N}$,*

(3) *M, when successively fed $f(0)$, $f(1)$, ... performs at most c ($c = *$ means at most finitely many) mind changes, i.e.,* $\mathrm{card}(\{n \mid M(f^n) \neq M(f^{n+1})\}) \leq c$.

If $(S_0, \ldots, S_{k-1})$ is classified by a CM M with at most c mind changes, then we write $(S_0, \ldots, S_{k-1}) \in CL^c(M)$. By CL_k^c we denote the collection of all k–tuples of sets that are classifiable with at most c mind changes, i.e.,

$$CL_k^c = \{(S_0, \ldots, S_{k-1}) \mid \exists \text{CM } M \; [(S_0, \ldots, S_{k-1}) \in CL_k^c(M)]\}.$$

Moreover, we set $CL^c = \bigcup_{k \geq 2} CL_k^c$. If $c = *$, then we usually omit the upper index, and say simply that $(S_0, \ldots, S_{k-1})$ is classifiable. Furthermore, if $c = 0$, then we also

say that $(S_0, \ldots, S_{k-1})$ is *finitely classifiable*, and set $FCL_k = CL_k^0$. In other words, a k–tuple $(S_0, \ldots, S_{k-1})$ is finitely classifiable if M's first guess is always a correct one. Finally, we set $FCL = \bigcup_{k \geq 2} FCL_k$.

In the definition above Requirement (1) specifies the set of allowed hypotheses. Clearly, any guess not contained in $\{0, \ldots, k-1\}$ cannot be correct. Hence, it is only natural to restrict the hypothesis space to the set $\{0, \ldots, k-1\}$. Nevertheless, this requirement does not restrict the capabilities of CMs. This can be seen as follows. Suppose we are given a CM M classifying some k-tuple $(S_0, \ldots, S_{k-1})$ of sets that uses the set $\mathbb{N}$ as its hypothesis space. Then M may be converted into a CM $\hat{M}$ satisfying Requirement (1) such that $(S_0, \ldots, S_{k-1}) \in CL(\hat{M})$. On any input, the CM $\hat{M}$ simply simulates M. If M outputs a guess from $\{0, \ldots, k-1\}$ then $\hat{M}$ behaves thus. Otherwise, $\hat{M}$ suppresses M's output and requests the next input. It is obvious that $CL_k^c(M) = CL_k^c(\hat{M})$. Thus, when dealing with classification, class preservedness (cf. Jantke and Beick (1981)) can be realized without loss of generality. On the other hand, looking at IIMs the situation changes considerably. The analog of Requirement (1) reads as "for all $f \in U$, all $n \in \mathbb{N}$, whenever M on input f^n outputs a hypothesis j, then $\varphi_j \in U$." However, the demand to work class preservingly does seriously restrict the learning power of IIMs (cf. Jantke and Beick (1981)).

In the next definition we explicitly consider the situation where the sets $S_0, ..., S_{k-1}$ are not necessarily disjoint. Looking at potential applications, it might be highly desirable not to obtain only *one* index of a set the target function f belongs to, but the indices of *all* those sets that contain f. For example, consider the case of automated medical diagnosis. In this case, we would certainly desire to be aware of *all* the diseases that manifest the observed symptoms. In our formalism, each disease is represented by a set S_i and the functions to be classified map symptoms to *present* or *not present*.

Definition 3. *Let $k \geq 2$, $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$ and let $S = S_0 \cup \ \ldots \ \cup S_{k-1}$. Furthermore, let $ALL = \{0, \ldots, k-1\}$. Then $(S_0, \ldots, S_{k-1})$ is said to be multi–classifiable iff there is a CM M such that for all $f \in S$,*

(1) *for all $n \in \mathbb{N}$, whenever M, on input f^n, outputs a hypothesis HYP, then $HYP \subseteq ALL$,*

(2) *there is a non-empty set $SUB \subseteq ALL$ such that*
   (a) *$M(f^n) = SUB$ for almost all n,*
   (b) *$f \in S_j$ for all $j \in SUB$,*
   (c) *$f \notin S_m$ for all $m \in ALL \setminus SUB$.*

If $(S_0, \ldots, S_{k-1})$ is multi–classified by a CM M, then we write $(S_0, \ldots, S_{k-1}) \in$ *Multi-CL*(M). By *Multi-CL*$_k$ we denote the collection of all k–tuples of sets that are multi–classifiable. Furthermore, we set *Multi-CL* $= \bigcup_{k \geq 2}$ *Multi-CL*$_k$. Analogously, *finite* multi-classification, *Multi-FCL*, can be defined.

Finally, we introduce *consistent* classification. The main intention is as follows. In general, potential users of a CM M never know whether M has successfully finished

its classification task. Hence, at least they might want to be sure that the actual hypothesis they receive is "as good as possible" in that it correctly reflects the information the CM has received so far. Therefore we require that every hypothesis has to describe a class always containing a function coinciding with the function to be classified on the whole initial segment given so far.

Definition 4. *Let $k \geq 2$, $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$ and $S = S_0 \cup \ldots \cup S_{k-1}$. Then $(S_0, \ldots, S_{k-1})$ is said to be consistently classifiable iff there is a CM M such that*

(1) $(S_0, \ldots, S_{k-1}) \in CL(M)$,

(2) *for all $f \in S$ and for all $n, i \in \mathbb{N}$, if $M(f^n) = i$, then there must be a function $g \in S_i$ such that $g =_n f$.*

If $(S_0, \ldots, S_{k-1})$ is consistently classifiable by a CM M then we write $(S_0, \ldots, S_{k-1}) \in$ *Cons-CL*(M). By *Cons-CL*$_k$ we denote the collection of all k–tuples of sets that are consistently classifiable. Finally, we set *Cons-CL* $= \bigcup_{k \geq 2}$ *Cons-CL*$_k$.

3. Classification of Predicates versus Classification of Arbitrary Functions

In this section we compare the classification of $\{0,1\}$ valued functions with the classification of functions in general. Looking at learning there are several results establishing major differences between the learnability of classes of predicates and the inferability of arbitrary classes of recursive functions (cf. Blum and Blum (1975), Zeugmann (1983, 1988), Osherson, Stob and Weinstein (1986)). Moreover, Freivalds, Kinber and Wiehagen (1995) discovered that consistent learning of predicates differs considerably from consistent identification of arbitrary recursive functions. Hence, it is only natural to ask whether there are differences also between the classification of predicates and arbitrary recursive functions.

Clearly, if a collection of sets of $\{0,1\}$ valued functions is classifiable, then it is classifiable as a collection of sets of arbitrary recursive functions. We now consider the converse direction.

Theorem 1. *There are pairwise disjoint sets S_0, S_1, S_2 such that*

(1) $S_0 \cup S_1 \cup S_2 = \mathcal{R}$,

(2) $(S_0, S_1, S_2) \notin CL$,

(3) $(\rho(S_0), \rho(S_1), \rho(S_2)) \in CL^2$.

Proof. The desired splitting of $\mathcal{R}$ is defined as follows. Let $S_0 = \{f \mid f \in \mathcal{R},$ *range* f is infinite$\}$, and let $S_1 = \{f \mid f \in \mathcal{R},$ *range* f is finite, $\forall x[f(x) = \max$ *range* $f \rightarrow f(x+1) \neq f(x)]\}$, and $S_2 = \mathcal{R} \setminus (S_0 \cup S_1)$. Obviously, $\mathcal{R} = S_0 \cup S_1 \cup S_2$ and S_0, S_1, and S_2 are pairwise disjoint. It remains to show that Assertions (2) and (3) are fulfilled.

Claim 1. $(S_0, S_1, S_2) \notin CL$.

Suppose the converse, i.e., there is a CM M such that $(S_0, S_1, S_2) \in CL(M)$. We continue with the definition of a recursive function f on which M fails. Define $f(0), f(1), \ldots$ to be zero until the least k is found such that M, when successively fed $f(0), \ldots, f(k)$, outputs its first hypothesis. Let $j = M(f^k)$. Now we distinguish between the following cases.

Case 1. $j = 0$ or $j = 1$.

Then set $f(k+1) = 0$.

Case 2. $j = 2$.

Then define $f(k+1) = 1$.

We proceed inductively as follows. Let k be the largest element for which f is already defined. Compute $j = M(f^k)$. We consider the following cases.

Case 1. $j = 0$.

Then define $f(k+1) = 0$.

Case 2. $j = 1$.

Then set $f(k+1) = \max\{f(y) \mid y \leq k\}$.

Case 3. $j = 2$.

Then define $f(k+1) = \max\{f(y) \mid y \leq k\} + 1$.

Obviously, f is recursive. Therefore, there has to be $j \in \{0, 1, 2\}$ such that $M(f^n) = j$ for almost all n.

Case 1. $j = 0$.

Due to the definition of f we obtain that $f(x) = 0$ for almost all $x \in \mathbb{N}$. Hence, f has a finite range and therefore it does not belong to S_0, a contradiction.

Case 2. $j = 1$.

In accordance with our construction, *range* f is finite. However, $f(x) = \max\{f(y) \mid y \in \mathbb{N}\}$ for almost all $x \in \mathbb{N}$. Therefore, M again misclassifies f.

Case 3. $j = 2$.

By definition of f, *range* f is infinite. Hence, f belongs to S_0 and not to S_2. Consequently, M again fails to correctly classify f.

This proves Claim 1.

Claim 2. $(\rho(S_0), \rho(S_1), \rho(S_2)) \in CL^2$.

For seeing this, observe that $\rho(S_0) = \emptyset$. Moreover, $\rho(S_1) = \{f \mid f \in \mathcal{R}_{0,1} \setminus \{0^\infty\}, \forall x[f(x) = 1 \rightarrow f(x+1) = 0]\}$ and $\rho(S_2) = \{f \mid f \in \mathcal{R}_{0,1}, \exists x[f(x) = f(x+1) = 1]\} \cup \{0^\infty\}$. Now it is easy to define a CM M witnessing $(\rho(S_0), \rho(S_1), \rho(S_2)) \in CL^2$. The machine M initially conjectures "2" until (if ever) an input $f(x)$ with $f(x) = 1$ is received. This is the correct behavior on the constant zero function. If an input 1

is received, then the input is not the constant zero function, and M conjectures "1." M continues scanning its input looking for two consecutive inputs that are 1. If this happens, then M changes its conjecture back to "2" and stops. ☒

The theorem has the following corollary.

Corollary 2. *For every $n \geq 3$, there is a collection of n pairwise disjoint sets exhausting $\mathcal{R}$ that is not in CL_n, but the collection of their restrictions to predicates is in CL_n.*

Proof. The idea is to divide S_1 from the proof of Theorem 1 into $n-2$ sets. These are the sets of recursive functions with finite range where the largest element of the range occurs at most once in a row, at most twice in a row, at most three times in a row, etc. The proof of Theorem 1 is then easily extended. ☒

The latter results show that there might be interesting differences between the classification of predicates and arbitrary functions. Later on, we shall point out some more differences. In the next section we study the power of classification algorithms with respect to the number of mind changes allowed.

4. Finite Classification versus Classification and Learnability

Our next theorem states that even the classification of two sets might be too complex to be done by a finitely working CM. Furthermore, the anticipated distinction between classification and finite classification is manifested.

Theorem 3. *There are nonempty, disjoint sets S_0, S_1 such that*

(1) $S_0 \cup S_1 = \mathcal{R}_{0,1}$,

(2) $(S_0, S_1) \in CL \setminus FCL$.

Proof. Let S_0 be the class containing only the function 0^∞. To immediately satisfy (1) and (2) above, let $S_1 = \mathcal{R}_{0,1} \setminus S_0$. A CM M that classifies (S_0, S_1) initially outputs "0" and then waits for a non zero input. When, and if, this input arrives, M changes its mind to "1." Clearly, $(S_0, S_1) \in CL(M)$.

Suppose by way of contradiction that $\hat{M}$ is a CM such that $(S_0, S_1) \in FCL(\hat{M})$. Then there must be an n such that $\hat{M}$, on input 0^n, outputs "0" as its conjectured classification. If not, then $\hat{M}$ fails to classify 0^∞ as a member of S_0, a contradiction. On the other hand, $\hat{M}$ outputs, when successively fed 0^n1^∞, the hypothesis "0" as its first guess. Therefore, $\hat{M}$ cannot finitely distinguish 0^∞ from 0^n1^∞. Consequently, $\hat{M}$ fails to properly classify at least one of those functions. ☒

The above theorem has an easy extension to the following:

Theorem 4. *For every $k \geq 2$, there are nonempty, pairwise disjoint sets $S_0, \ldots, S_{k-1}$ such that*

(1) $S_0 \cup S_1 \cup \ldots \cup S_{k-1} = \mathcal{R}_{0,1}$,

(2) $(S_0, S_1, \ldots, S_{k-1}) \in CL \setminus FCL$.

Proof. Let $k \geq 2$ be arbitrarily fixed. For $0 \leq i < k-1$, S_i is the subset of $\mathcal{R}_{0,1}$ that contains all functions f such that the cardinality of the set $\{x \mid f(x) \neq 0\}$ is exactly i. Moreover, we set $S_{k-1} = \mathcal{R}_{0,1} \setminus (S_0 \cup \ldots \cup S_{k-2})$. Clearly, (1) above is satisfied. A CM M that classifies $S_0, S_1, \ldots, S_{k-1}$ initially outputs "0" and then reads input, counting the number of nonzero inputs. After a nonzero input is observed, resulting in a total of j nonzero inputs, M outputs the minimum of $\{j, k-1\}$. Clearly, $(S_0, S_1, \ldots, S_{k-1}) \in CL(M)$.

Suppose by way of contradiction that M' is a CM such that $(S_0, S_1, \cdots, S_{k-1}) \in FCL(M')$. As in Theorem 3, there must be an n such that M', on input 0^n produces a conjecture "0." Then M' will improperly classify at least one of 0^∞ or 0^n10^∞. ☒

In contrast to Theorem 4, it is possible to split $\mathcal{R}_{0,1}$ into disjoint, finitely classifiable sets.

Theorem 5. *For every $k \geq 2$, there are nonempty, pairwise disjoint sets $S_0, \ldots, S_{k-1}$ such that*

(1) $S_0 \cup \ldots \cup S_{k-1} = \mathcal{R}_{0,1}$,

(2) $(S_0, \ldots, S_{k-1}) \in FCL$.

Proof. Let $S_0 = \{f \mid f \in \mathcal{R}_{0,1} \text{ and } f(0) = 0\}$. For $1 \leq i < k$, $S_i = \{f \mid f \in \mathcal{R}_{0,1}, f(j) = 1 \text{ for } 0 \leq j < i \text{ and } f(i) = 0\}$. Finally, $S_{k-1} = \mathcal{R}_{0,1} \setminus (S_0 \cup S_1 \cup \ldots \cup S_{k-2})$. Clearly, (1) above holds. A CM M that finitely classifies $(S_0, S_1, \ldots, S_{k-1})$ starts by reading input until the values of $f(0)$, $f(1)$, ..., $f(k-2)$ are all known. If all of these values are 1, then output "$k-1$" as the only conjecture. If not, then output "i" where i is the least number such that $f(i) = 0$. ☒

Next we compare finite classification and learnability.

Theorem 6. *For all $S_0 \subseteq \mathcal{R}_{0,1}$ such that $S_0 \in FIN$, there is a nonempty $S_1 \subseteq \mathcal{R}_{0,1}$ such that*

(1) $S_1 \in FIN$,

(2) $S_0 \cap S_1 = \emptyset$,

(3) $(S_0, S_1) \in FCL$.

Proof. Suppose $S_0 \subseteq \mathcal{R}_{0,1}$ and $S_0 \in FIN$. Since no set in FIN can contain an accumulation point (cf. Freivalds and Wiehagen (1979)), there is $\alpha \in \{0,1\}^*$ such that no function in $\mathcal{R}_{0,1}$ extending α is in S_0. Let $S_1 = \{\alpha 0^\infty\}$. Clearly, $S_1 \subseteq \mathcal{R}_{0,1}$ and $S_1 \in FIN$. By the choice of α, $S_0 \cap S_1 = \emptyset$. A CM M witnessing $(S_0, S_1) \in FCL$ knows α. Let $\alpha = (\alpha_0, \ldots, \alpha_n)$. Then M requests more and more inputs until it has seen $(f(0), \ldots, f(n))$. In case $f^n = \alpha$ it outputs "1" and "0" otherwise. Clearly, $(S_0, S_1) \in FCL(M)$. ☒

Theorem 6 allows the following interpretation. Any "easily" learnable class can always be paired with an "easily" learnable class to form an "easily" classifiable pair. Thus, sometimes learning and classification are *not* very distinct. On the other hand, we have the following result.

Theorem 7. *For all $S_0 \subseteq \mathcal{R}_{0,1}$ such that $S_0 \in FIN$, there is a nonempty $S_1 \subseteq \mathcal{R}_{0,1}$ such that*

(1) $S_1 \notin EX$,

(2) $S_0 \cap S_1 = \emptyset$,

(3) $(S_0, S_1) \in FCL$.

Proof. Let α as in the proof of Theorem 6. Set $S_1 = \{\alpha f \mid f \in \mathcal{R}_{0,1}\}$. A CM witnessing $(S_0, S_1) \in FCL$ may work as described in the proof of Theorem 6. It remains to show that $S_1 \notin EX$. Suppose the converse, i.e., there is an IIM M such that $S_1 \in EX(M)$. Then any program for M can be used to construct an IIM $\hat{M}$ that infers $\mathcal{R}_{0,1}$, a contradiction (cf. Gold (1965)). This can be seen as follows. On any input f^n, $n \in \mathbb{N}$, the IIM $\hat{M}$ simulates M on input αf^n. Suppose $j = M(\alpha f^n)$. Then $\hat{M}$ outputs a canonical program of the function η defined as follows: $\eta(x) = \varphi_j(x+n)$ for all $x \in \mathbb{N}$. Clearly, $\eta = f$ if $\varphi_j = \alpha f$. On the other hand, since $S_1 \in EX(M)$, there is a j such that $\varphi_j = \alpha f$ and $M(\alpha f^n) = j$ for almost all n. Consequently, $\hat{M}$ learns f. ☒

5. Bounding the Number of Mind Changes

For finite classification (FCL) a CM is not allowed to change its conjecture at all. For standard classification (CL) a CM is allowed to change its conjecture an arbitrary finite number of times. The precise number of mind changes does not have to be determined in advance. In the study of inductive inference, a mind change hierarchy was discovered by fixing in advance a particular number of mind changes that an IIM was allowed to make (cf. Case and Smith (1983)). In the sequel we present a hierarchy for classification based on a fixed number of allowed mind changes.

Theorem 8. *For every $k \in \mathbb{N}$, there are nonempty, disjoint sets S_0, S_1, ..., S_{k+1} such that*

(1) $S_0 \cup S_1 \cup \ldots \cup S_{k+1} = \mathcal{R}_{0,1}$,

(2) $(S_0, \ldots, S_{k+1}) \in CL^{k+1} \setminus CL^k$.

Proof. Suppose k is given. For $i \leq k$, let S_i be the subset of $\mathcal{R}_{0,1}$ such that for any $f \in S_i$ the cardinality of $\{x \mid f(x) = 1\}$ is exactly i. For example, $S_0 = \{0^\infty\}$. Finally, $S_{k+1} = \mathcal{R}_{0,1} \setminus (S_0 \cup S_1 \cup \ldots \cup S_k)$. Clearly, (1) above is satisfied.

A CM that witnesses $(S_0, \ldots, S_{k+1}) \in CL^{k+1}$ may work as follows: Let f be any fixed predicate. The CM starts by conjecturing "0". If the most recent conjecture is "i," $i \leq k$ and another datum $f(x)$ that is 1 is observed as input, then the CM conjectures "$i+1$." When the CM produces a conjecture "i" it will constitute the i^{th} change of conjecture. Clearly, the CM described above is the desired witness.

The proof is completed by showing that $(S_0, \ldots, S_k) \notin CL^k$. Suppose by way of contradiction that M is a CM such that $(S_0, \ldots, S_k) \in CL^k(M)$. A finite $\{0,1\}$ valued function is constructed in at most $k+1$ steps below. It will turn out that some extension of this finite function will be an $f \in \mathcal{R}_{0,1}$ that M cannot properly classify. The finite initial segment constructed prior to Step s is denoted by the string σ^s. By way of initialization, σ^0 is the empty string. An invariant over Steps $0 \leq s \leq k$ is that after Step s has been completed, the number of occurrences of "1" in σ^{s+1} is exactly s and M, when successively fed σ^{s+1}, will make at least s mind changes.

> *Step* 0. Look for the least n such that M on input 0^n produces a conjecture. If such an n is found, set $\sigma^1 = 0^n$ and go to Step 1. If no such n exists, then M fails to correctly classify the everywhere zero function in S_0.

> *Step* s, $0 < s \leq k$. Let σ^s be the initial segment already defined. Look for the first pair (n, m) in the Cantor enumeration of $\mathbb{N} \times \mathbb{N}$ such that $M(\sigma^s) \neq M(\sigma^s 0^m 10^n)$. In other words, we seek for one more mind change of M on some additional data $0^m 10^n$. If such an m and n are found then set $\sigma^{s+1} = \sigma^s 0^m 10^n$ and go to Step $s+1$ until $s+1 > k$.

First we prove by induction the invariant that after Step s has been completed, the number of occurrences of "1" in σ^{s+1} is exactly s and M, when successively fed σ^{s+1}, will make at least s mind changes. After Step 0 has been completed, σ^1 has no ones present. Moreover, M, on input σ^1, outputs its first guess. Suppose inductively that for $s < k$, after Step s has been completed, the number of occurrences of "1" in σ^{s+1} is exactly s and M, when successively fed σ^{s+1}, will make at least s mind changes. After Step $s+1$ is complete, exactly one more "1" has been added to the string σ^{s+1}, and M has been forced to make at least one more mind change.

By the comments in Step 0, we know that Step 0 must be completed, as otherwise M fails to classify the function 0^∞. Suppose Step $s < k$ is the last step completed. Let $f_1 = \sigma^{s+1}0^\infty$ and $f_2 = \sigma^{s+1}10^\infty$. By the invariant over steps, $f_1 \in S_s$ and $f_2 \in S_{s+1}$. By the failure of Step $s+1$ to complete, M will classify both f_1 and f_2 as in the same set, a contradiction.

To complete the proof, suppose Step k completes. Let $f_1 = \sigma^{k+1}0^\infty$ and $f_2 = \sigma^{k+1}10^\infty$. By the invariant over steps, $f_1 \in S_k$ and $f_2 \in S_{k+1}$. M will make its full allotment of mind changes on the initial segment σ^{k+1}. Hence, M will classify both f_1 and f_2 as in the same set, a contradiction. ☒

Moreover, it is also possible to prove a hierarchy in the number of mind changes that is not related to the number of sets to be classified.

Theorem 9. *For all $c \in \mathbb{N}$, there are nonempty, disjoint sets $S_0, S_1 \subseteq \mathcal{R}_{0,1}$ such that $(S_0, S_1) \in CL^{c+1} \setminus CL^c$.*

Proof. Let $c \in \mathbb{N}$ be arbitrarily fixed. Furthermore, let g be a $\{0,1\}$ valued recursive function of two arguments satisfying

(1) for all $i \in \mathbb{N}$, $g(i,n) = 0$ for almost all $n \in \mathbb{N}$ or $g(i,n) = 1$ for almost all $n \in \mathbb{N}$,

(2) for all $i \in \mathbb{N}$, $\mathrm{card}(\{n \mid n \in \mathbb{N},\ g(i,n) \neq g(i,n+1)\}) \leq c+1$,

(3) there is no recursive function h such that

(3.1) $h(i,n) = g(i,n)$ for all $i \in \mathbb{N}$ and almost all $n \in \mathbb{N}$,

(3.2) $\mathrm{card}(\{n \mid n \in \mathbb{N},\ h(i,n) \neq h(i,n+1)\}) \leq c$ for all $i \in \mathbb{N}$.

In other words, g is a $\{0,1\}$ valued limiting recursive function that can be computed with at most $c+1$ mind changes but no limiting recursive function can compute the limits of g with at most c mind changes. The existence of such function g has been proved by Gold (1965).

Now we define $S_0, S_1 \subseteq \mathcal{R}_{0,1}$ as follows: $S_0 = \{0^i 1^\infty \mid i \in \mathbb{N}$ and $g(i,n) = 0$ for almost all $n\}$, and $S_0 = \{0^i 1^\infty \mid i \in \mathbb{N}$ and $g(i,n) = 1$ for almost all $n\}$.

By definition, $S_0 \cap S_1 = \emptyset$. A CM M witnessing $(S_0, S_1) \in CL^{c+1}$ first counts the number of consecutive zeros until it receives the first 1, and outputs nothing. Then, on any input $0^i 1^n$ it computes and outputs $g(i, i+n)$. In accordance with the choice of function g it directly follows that $(S_0, S_1) \in CL^{c+1}(M)$. On the other hand, it is easy to see that any CM $\hat{M}$ with $(S_0, S_1) \in CL^c(\hat{M})$ may be used to design a program for a recursive function h satisfying (3.1) and (3.2), a contradiction to the choice of g. Hence, the theorem follows. ☒

6. Classification versus Multi–Classification

In this and in the next section we compare the power of classification, multi–classification and consistent classification. In particular, we are interested in learning whether or not multi–classification or consistent classification is harder to achieve than ordinary classification. Since neither multi–classification nor consistent classification has been studied in the framework presented here, our goal is twofold. First we are interested in general results concerning the classification power of these models. Second we ask whether or not the results obtained extend to the classification of predicates.

Our next theorem compares multi–classification and ordinary classification in both the finite and the limit cases.

Theorem 10.

(1) *Multi-FCL* $\subset$ *FCL.*

(2) *Multi-CL* $\subset$ *CL.*

Proof. Clearly, *Multi-FCL* $\subseteq$ *FCL* as well as *Multi-CL* $\subseteq$ *CL*. In order to show that these inclusions are proper we use the following simple set theoretic argument. Suppose there are two sets S_0 and S_1 such that $(S_0, S_1) \notin CL$. Hence, $(S_0, S_1) \notin FCL$. Then $(S_0, S_0 \cup S_1) \in FCL$ is witnessed by a classification machine that always selects $S_0 \cup S_1$. However, if $(S_0, S_0 \cup S_1)$ were in *Multi-CL* then it would be possible to classify (S_0, S_1) in the limit, a contradiction. Thus, $(S_0, S_0 \cup S_1) \notin$ *Multi-FCL*, too.

Consequently, it remains to show that there are sets S_0 and S_1 satisfying $(S_0, S_1) \notin CL$. Let $S_0 = \{f \in \mathcal{R} \mid$ *range* f is finite$\}$, and let $S_1 = \{f \in \mathcal{R} \mid$ *range* f is infinite$\}$. Using the same technique as in the proof of Theorem 1 one easily verifies that $(S_0, S_1) \notin CL$. This completes the proof. ☒

Applying a simple coding technique, the proof of Theorem 10 directly yields the following corollary.

Corollary 11. *There are sets* $S_0, S_1 \subseteq \mathcal{R}_{0,1}$ *such that* $(S_0, S_1) \in FCL \setminus$ *Multi-CL.*

As we have seen, an algorithmically solvable classification problem may become intractable if treated as multi-classification problem. If a problem is algorithmically unsolvable it is only natural to ask for its *degree of unsolvability*. As our next theorem shows the problem to finitely multi-classify two sets that are themselves finitely classifiable might become at least as hard as the halting problem. This nicely contrasts the result that the degree of unsolvability for $\mathcal{R} \in EX$ is strictly less than the degree of the halting problem (cf. Adleman and Blum (1991)).

Theorem 12. *There are nonempty sets* $S_0, S_1 \subseteq \mathcal{R}$ *such that*

(1) $(S_0, S_1) \in FCL \setminus$ *Multi-FCL,*

(2) *The degree of* $(S_0, S_1) \in$ *Multi-FCL is equivalent to the degree of the halting problem.*

Proof. Let $S_0 = \{f \mid f \in \mathcal{R},\ f(0) \in K\}$, and let $S_1 = \{f \mid f \in \mathcal{R},\ f(1) \in K\}$.

By definition, for any $f \in S_0 \cup S_1$, we have $f(0) \in K$ or $f(1) \in K$. Hence, a CM witnessing $(S_0, S_1) \in FCL$ may work as follows. After having received both $f(0)$ and $f(1)$ it tries to compute $\varphi_{f(0)}(f(0))$ and $\varphi_{f(1)}(f(1))$. One of these computations has to stop. In case the CM verifies $f(0) \in K$, it outputs "0." If the CM verifies $f(1) \in K$, then it outputs "1." Clearly, $(S_0, S_1) \in FCL(M)$.

By way of contradiction assume that $(S_0, S_1) \in \mathit{Multi\text{-}FCL}(M)$ for some CM M. Then the halting set K is decidable by the following procedure.

On input $z \in \mathbb{N}$ the procedure chooses an arbitrary $k \in K$. Then it uses M to finitely multi–classify $f = kz0^\infty$. The procedure feeds successively $f(0), f(1), \ldots$ to M until M produces its hypothesis. If M outputs "0" then the procedure outputs "$z \notin K$." In case M's output is "$(0,1)$" the procedure outputs "$z \in K$."

It remains to show that the procedure defined above does work correctly. First, since $f(0) \in K$, the function f belongs to $S_0 \cup S_1$. Hence, M must finitely multi–classify f. Therefore, when fed $f(0), f(1), \ldots$ the CM M has to output a correct hypothesis. The rest directly follows from the definition of S_0 and S_1. This proves (1).

As shown above, any CM M witnessing $(S_0, S_1) \in \mathit{Multi\text{-}FCL}(M)$ can be effectively transformed into an algorithm that solves the halting problem. The other direction is also true. Given an oracle for the halting problem one straightforwardly can construct a CM M that multi-classifies (S_0, S_1). ☒

The latter theorem gives an example of sets that are classifiable but not multi-classifiable. Using a similar proof technique, it is possible to give another such example where the sets are chosen from *NUM*. All the sets in *NUM* are known to be easy to learn, Gold (1967). On the other hand, Barzdin (1971) showed that *NUM* is a *proper* subset of *EX*.

Theorem 13. *There are sets $S_0, S_1 \in \mathit{NUM}$ such that $(S_0, S_1) \in \mathit{FCL} \setminus \mathit{Multi\text{-}FCL}$.*

Proof. Let U be the set of all functions $\alpha 0^\infty$ as α ranges over the string representations of finite functions with domain an initial segment of $\mathbb{N}$. Clearly, $U \in \mathit{NUM}$. Let $S_0 = \{f \mid f \in U,\ f(0) \in K\}$ and $S_1 = \{f \mid f \in U,\ f(1) \in K\}$. Obviously, $S_0, S_1 \in \mathit{NUM}$. The rest of the proof is similar to the proof of Theorem 12. ☒

If in the proof of Theorem 13 we used, instead of K, some set that was limit r.e. but not limiting recursive (cf. Gold, 1965), then $(S_0, S_1) \in \mathit{CL} \setminus \mathit{Multi\text{-}CL}$ would still hold.

We now want to derive two sufficient conditions for pairs of *NUM*–classes being multi–classifiable. The first condition uses the decidability of program equivalence with respect to particular numberings (cf. Theorem 14). The second condition, given in Theorem 15, is a topological one using the notion of accumulation points.

Let $S_0, S_1 \in \mathit{NUM}$. Let $\psi, \tau \in \mathcal{R}^2$ be numberings such that $S_0 = \mathcal{R}_\psi$ and $S_1 = \mathcal{R}_\tau$. We say that (ψ, τ)-equivalence is decidable iff there is a recursive predicate *dec* of two arguments such that for any $i, j \in \mathbb{N}$, $\mathit{dec}(i,j) = 1$ if and only if $\psi_i = \tau_j$.

Theorem 14. *Let $S_0, S_1 \in \mathit{NUM}$. Let $\psi, \tau \in \mathcal{R}^2$ be numberings such that $S_0 = \mathcal{R}_\psi$, $S_1 = \mathcal{R}_\tau$ and (ψ, τ)-equivalence is decidable. Then $(S_0, S_1) \in \mathit{Multi\text{-}CL}$.*

Proof. First we want to describe the idea of the proof informally. Let $f \in S_0 \cup S_1$. Then the corresponding CM M first tries to learn(!) the function f with respect to both numberings ψ and τ. After succeeding with respect to, say ψ, i.e., after finding

out in the limit some $i \in \mathbb{N}$ such that $\psi_i = f$, M using the decidability of (ψ, τ)-equivalence tries to find a $j \in \mathbb{N}$ such that $\psi_i = \tau_j$ in which case M converges to the multi–hypothesis (0,1); otherwise, M converges to the single hypothesis 0.

In order to formalize this idea let $S_0, S_1 \in NUM$, let $\psi, \tau \in \mathcal{R}^2$ be numberings such that $S_0 = \mathcal{R}_\psi$, $S_1 = \mathcal{R}_\tau$, and let dec decide (ψ, τ)-equivalence.

Then define E_ψ as a slightly modified version of identification by enumeration, cf. Gold (1967), as follows.

$$E_\psi(f^n) = \begin{cases} \min \mathrm{I}, & \text{if } \mathrm{I} = \{i \mid i \leq n, \psi_i =_n f\} \text{ and } \mathrm{I} \neq \emptyset, \\ n, & \text{if } \mathrm{I} = \emptyset. \end{cases}$$

Clearly, if $f \in \mathcal{R}_\psi$, then $E_\psi(f^n) = \min\{i \mid \psi_i = f\}$ for almost all $n \in \mathbb{N}$. E_τ is defined analogously.

Now define a CM M as follows.

$$M(f^n) = \begin{cases} (0,1), & \text{if } dec\,(E_\psi(f^n), E_\tau(f^n)) = 1, \\ 0, & \text{if } dec\,(E_\psi(f^n), E_\tau(f^n)) = 0 \text{ and } E_\psi(f^n) \leq E_\tau(f^n), \\ 1, & \text{if } dec\,(E_\psi(f^n), E_\tau(f^n)) = 0 \text{ and } E_\psi(f^n) > E_\tau(f^n). \end{cases}$$

Let $f \in S_0 \cup S_1$.

Case 1. $f \in S_0 \cap S_1$.

Then, by the property of E above, there are $i, j \in \mathbb{N}$ such that $\psi_i = f$, $\tau_j = f$ (hence $dec(i,j) = 1$), and for almost all $n \in \mathbb{N}$, $E_\psi(f^n) = i$ and $E_\tau(f^n) = j$. Consequently, for almost all $n \in \mathbb{N}$, $M(f^n) = (0,1)$.

Case 2. $f \in S_0$, $f \notin S_1$.

Then there is an $i \in \mathbb{N}$ such that $\psi_i = f$ and for almost all $n \in \mathbb{N}$, $E_\psi(f^n) = i$. Since $f \notin S_1$, $dec(i, E_\tau(f^n)) = 0$ for any $n \in \mathbb{N}$. Moreover, since $f \notin S_1$, the sequence $(E_\tau(f^n))_{n \in \mathbb{N}}$ is unbounded. Hence, for almost all $n \in \mathbb{N}$, $i = E_\psi(f^n) < E_\tau(f^n)$. Consequently, for almost all $n \in \mathbb{N}$, $M(f^n) = 0$.

Case 3. $f \notin S_0$, $f \in S_1$.

Analogous to Case 2.

Hence $(S_0, S_1) \in$ *Multi-CL* by M. ☒

Observe, that the classification machine from the proof of Theorem 14 uses a learning algorithm, namely identification by enumeration, as a subroutine. An example for the converse, i.e., an inductive inference machine using a classification algorithm as a subroutine, can be found in Wiehagen and Smith (1995). We now present another sufficient condition for r.e. classes being multi–classifiable.

Theorem 15. *Let* $S_0, S_1 \in NUM$. *Let* $(S_0 \setminus S_1) \cap Acc(S_1) = (S_1 \setminus S_0) \cap Acc(S_0) = \emptyset$. *Then* $(S_0, S_1) \in$ *Multi-CL.*

Proof. The idea is the following. In order to multi–classify $f \in S_0 \cup S_1$ a CM M now may confine itself to search only for "candidates" for f within the corresponding numberings ψ, τ of $S_0,\ S_1$, respectively. A function, say ψ_i, is a candidate for f on f^n iff $\psi_i =_n f$. Suppose that $f \in S_0$ but $f \notin S_1$. Then $(S_0 \setminus S_1) \cap Acc(S_1) = \emptyset$ guarantees that for almost all $n \in \mathbb{N}$, within τ there will be even no candidate for f. This enables M to converge to the correct hypothesis 0.

We now proceed formally. Let $S_0, S_1 \in NUM$. Let $(S_0 \setminus S_1) \cap Acc(S_1) = (S_1 \setminus S_0) \cap Acc(S_0) = \emptyset$. Let $\psi, \tau \in \mathcal{R}^2$ be numberings such that $S_0 = \mathcal{R}_\psi$ and $S_1 = \mathcal{R}_\tau$. Then define a CM M as follows.

$$M(f^n) = \begin{cases} (0,1), & \text{if } \psi_i =_n f \text{ and } \tau_j =_n f \text{ for some } i, j \leq n, \\ 0, & \text{if } \psi_i =_n f \text{ for some } i \leq n, \text{ and } \tau_j \neq_n f \text{ for all } j \leq n, \\ 1, & \text{if } \psi_i \neq_n f \text{ for all } i \leq n, \text{ and } \tau_j =_n f \text{ for some } j \leq n. \end{cases}$$

Let $f \in S_0 \cup S_1$.

Case 1. $f \in S_0 \cap S_1$.

Then, for almost all $n \in N,\ M(f^n) = (0,1)$.

Case 2. $f \in S_0, f \notin S_1$.

Since $(S_0 \setminus S_1) \cap Acc(S_1) = \emptyset$, there is an $m \in \mathbb{N}$ such that for any $n \geq m$ and any $j \in \mathbb{N}, \tau_j \neq_n f$. Consequently, for all $n \geq m, M(f^n) = 0$.

Case 3. $f \notin S_0, f \in S_1$.

Then, analogously, for almost all $n \in \mathbb{N},\ M(f^n) = 1$.

Hence, $(S_0, S_1) \in$ *Multi-CL* by M. ☒

Note that in contrast to Theorem 14 the condition of Theorem 15 is independent of the choice of any numberings of the classes to be multi–classified. However, the conditions of Theorems 14 and 15 turn out to be not necessary. We now show this for the condition of Theorem 15.

Proposition 16. There are $S_0, S_1 \in NUM$ such that

(1) $(S_0 \setminus S_1) \cap Acc(S_1) \neq \emptyset$,

(2) $(S_0, S_1) \in$ *Multi-CL*.

Proof. Let $S_0 = \{0^i 1^\infty \mid i \in \mathbb{N}\} \cup \{0^\infty\}$ and $S_1 = S_0 \setminus \{0^\infty\}$. Then $S_0 \setminus S_1 = \{0^\infty\}$ and $0^\infty \in Acc(S_1)$. Hence $(S_0 \setminus S_1) \cap Acc(S_1) \neq \emptyset$. Now define a CM M as follows:

$M(0^i) = 0$ and $M(0^i 1^j) = (0,1)$.

Obviously, $(S_0, S_1) \in$ *Multi-CL* by M. ☒

It remains open whether or not a necessary and sufficient condition for multi–classifiability of *NUM*–classes can be found which essentially relies just on the recursive enumerability of the classes.

We now aim to characterize multi–classification of *arbitrary* classes in terms of standard classification. For that purpose, we introduce the notion of a *mosaic*. Let $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$ and $S = S_0 \cup \ldots \cup S_{k-1}$. Let $ALL = \{0, \ldots, k-1\}$ and $SUB \subseteq ALL$. Then let $S_{SUB} = \bigcap_{j \in SUB} S_j \setminus \bigcup_{i \in ALL \setminus SUB} S_i$. S_{SUB} is the set of all $f \in S$ that belong to all of the classes S_j, $j \in SUB$, and to none of the classes S_i, $i \in ALL \setminus SUB$. Finally, let $mosaic(S_0, \ldots, S_{k-1})$ be the tuple of all sets S_{SUB}, where $SUB \subseteq ALL$.

Obviously, $mosaic(S_0, \ldots, S_{k-1})$ is a 2^k–tuple of pairwise disjoint sets the union of which is S (cf. Figure 1 for the $k = 3$ case). Note that some components of $mosaic(S_0, \ldots S_{k-1})$ may be empty.

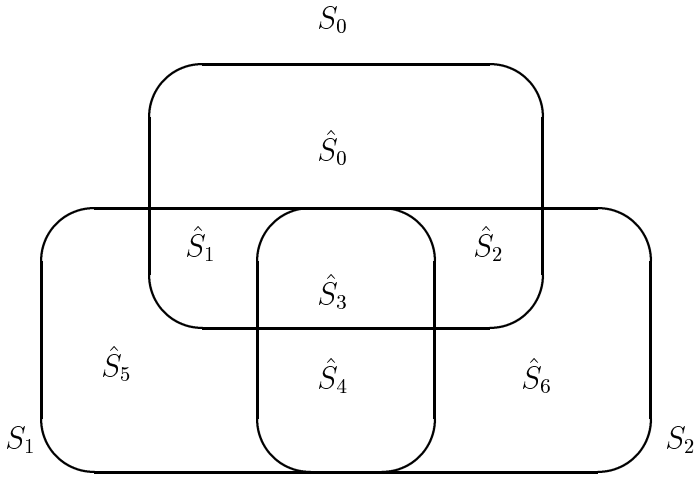

Figure 1: The mosaic $\hat{S}_0, \ldots, \hat{S}_6, \hat{S}_7$ of the sets S_0, S_1, S_2, where $\hat{S}_7 = \emptyset$.

Finally, let $mosaic(S_0, \ldots, S_{k-1}) = (T_0, \ldots, T_{2^k-1})$. Moreover, without loss of generality we assume that the T_j's, $0 \leq j \leq 2^k - 1$, are ordered in such a way that from the index j the corresponding set SUB can be computed such that $S_{SUB} = T_j$ and vice versa. Formally, this can be realized by a one-to-one mapping m from the set of all subsets SUB of ALL onto the set $\{0, \ldots, 2^k - 1\}$ such that for any $SUB \subseteq ALL$, $S_{SUB} = T_{m(SUB)}$. We have obtained the following characterizations.

Theorem 17. *For all $k \geq 2$ and all sets $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$, $(S_0, \ldots, S_{k-1}) \in$ Multi-CL if and only if $mosaic(S_0, \ldots, S_{k-1}) \in CL$.*

Proof. Necessity. Let $(S_0, \ldots, S_{k-1}) \in Multi\text{-}CL(M)$. Let $\hat{M}$ be a CM such that for any $f \in S_0 \cup \ldots \cup S_{k-1}$ and any $n \in \mathbb{N}$, $\hat{M}(f^n) = m(M(f^n))$. Clearly, $mosaic(S_0, \ldots, S_{k-1}) \in CL(\hat{M})$.

Sufficiency. Let $mosaic(S_0, \ldots, S_{k-1}) \in CL(M)$ for some CM M. Furthermore, let $\hat{M}$ be a CM such that for any $f \in S_0 \cup \ldots \cup S_{k-1}$ and any $n \in \mathbb{N}$, $\hat{M}(f^n) = m^{-1}(M(f^n))$. Obviously, $(S_0, \ldots, S_{k-1}) \in Multi\text{-}CL(\hat{M})$. ☒

Analogously, the following characterization of finite multi–classification can be proved.

Theorem 18. *For all $k \geq 2$ and all sets $S_0, \ldots, S_{k-1} \subseteq \mathcal{R}$, $(S_0, \ldots, S_{k-1}) \in$ Multi-FCL if and only if mosaic$(S_0, \ldots, S_{k-1}) \in$ FCL.*

Notice that Theorems 17 and 18 equates a multi-classification problem with a classification problem with exponentially many more possible classifications. To the extent that the number of possible classifications is regarded as complexity measure, Theorems 17 and 18 highlights the relative difficulty of classification and multi-classification.

7. Classification versus Consistent Classification

Intuitively, consistency is a desirable quality for both learning and classifying machines in that consistent hypotheses correctly and completely reflect the information sample seen so far. Moreover, any inconsistent hypothesis will definitely be incorrect. Hence it seems reasonable to confine both learning and classifying machines to produce consistent hypotheses, only. However, surprisingly, it turns out that consistent learners are of less power than arbitrary ones. Call an IIM M consistent iff for any function f from the class to be learned and any $n \in \mathbb{N}$, its hypothesis $M(f^n)$ describes a function g such that $g =_n f$. Clearly, any hypothesis not possessing this natural property of consistency would be incorrect. On the other hand, Barzdin (1974) showed that there are learnable classes of recursive functions which cannot be learned by any consistent IIM. Also, Blum and Blum (1975) observed that many powerful IIMs do not work consistently. Recently, Wiehagen and Zeugmann (1994) proved that the inconsistency phenomenon even remains valid in the world of learning *in polynomial time*.

The main goal of this section is to show that also for classification an analogous inconsistency phenomenon does hold. First we relate consistent classification of arbitrary recursive functions to both standard and finite classification.

Theorem 19. *FCL $\subset$ Cons-CL $\subset$ CL.*

Proof. The inclusion *FCL* $\subseteq$ *Cons-CL* is an immediate consequence of the corresponding definitions. In order to show that this inclusion is proper we can use the same ideas as in the proof of Theorem 8. Let $S_0 = \{0^\infty\}$, and let $S_1 = \{0^n 10^\infty \mid n \in \mathbb{N}\}$. Hence, $(S_0, S_1) \notin$ *FCL*. A CM M witnessing $(S_0, S_1) \in$ *Cons-CL* can be defined as follows. As long as M exclusively reads data that are 0 it conjectures "0." If it sees a datum that is 1, it changes its mind to "1." Obviously, M is consistent, and thus the first inclusion is proved.

Next, we define two disjoint sets S_0, S_1 splitting $\mathcal{R}$ such that $(S_0, S_1) \in CL^1 \setminus$ *Cons-CL*. Let $S_0 = \{f \mid f \in \mathcal{R},\ f(0) \in K\}$ and $S_1 = \{f \mid f \in \mathcal{R},\ f(0) \notin K\}$. Clearly, $S_0 \cap S_1 = \emptyset$ and $S_0 \cup S_1 = \mathcal{R}$. An CM M witnessing $(S_0, S_1) \in CL^1$ may work as follows. On any input f^n, $f \in \mathcal{R}$, $n \in \mathbb{N}$ the machine M tries to compute $\varphi_{f(0)}(f(0))$ within at most n steps. If the computation of $\varphi_{f(0)}(f(0))$ stops within this time bound then M outputs "0." Otherwise, it conjectures "1." Obviously,

$(S_0, S_1) \in CL^1(M)$.

It remains to show that $(S_0, S_1) \notin \mathit{Cons}\text{-}\mathit{CL}$. Suppose by way of contradiction that there is a CM M that consistently classifies (S_0, S_1). We now show that M can be used to construct effectively an algorithm that decides K, a contradiction.

The wanted algorithm $\mathcal{A}$ is defined as follows. On any input $z \in \mathbb{N}$ the algorithm $\mathcal{A}$ uses M to classify consistently $f = z^\infty$. More precisely, $\mathcal{A}$ simulates the computation of $M(f^0),\ M(f^1), \ldots$ until M, when successively fed $f(0),\ f(1), \ldots$ outputs its first guess j. If $j = 0$, then $\mathcal{A}$ outputs "$z \in K$." If $j = 1$, then $\mathcal{A}$ outputs "$z \notin K$."

Obviously, $\mathcal{A}$ is recursive. It remains to show that $\mathcal{A}$ works correctly. Suppose $j = 0$. In accordance with the definition of consistent classification there has to be a function $g \in S_0$ such that f and g do coincide up to n, where n is the least number such that M, when fed f^n outputs its first conjecture. In particular, $g(0) = f(0) = z$. Hence, $z \in K$ and $\mathcal{A}$ behaves correctly. Finally, suppose $j = 1$. Again there has to be a $g \in \mathcal{R}$ such that $g \in S_1$ and $g(0) = f(0) = z$. Hence, $z \notin K$. ☒

The next corollary shows that Theorem 19 remains valid for classifying predicates.

Corollary 20. *There are sets* $S_0,\ S_1 \subseteq \mathcal{R}_{0,1}$ *such that* $(S_0, S_1) \in CL \setminus \mathit{Cons}\text{-}\mathit{CL}$.

Proof. Let $S_0 = \{1^i 0^\infty \mid i \in K\}$ and $S_1 = \{1^i 0^\infty \mid i \notin K\}$. Then the same arguments as in the proof of Theorem 19 apply mutatis mutandis. We omit the details. ☒

Perhaps a classification procedure maintains consistency by procrastinating until it is sure. In real applications, it may not be tolerable for a classification procedure to produce no output while it is waiting for more data. To isolate this effect, we define a restricted version of the classification mechanisms. We say that a CM is *responsive* if it outputs a conjectured classification after every input. A simple argument shows that responsiveness poses no restrictions for standard and finite classification. However, combining responsiveness and consistency results in a severe restriction on classification capabilities, as our next theorem shows. Let $\mathit{ConsR}\text{-}\mathit{CL}$ denote the collection of n-tuples of sets, as n varies, that are classifiable by CM's which are both consistent and responsive.

Theorem 21.

(1) $\mathit{FCL} \setminus \mathit{ConsR}\text{-}\mathit{CL} \neq \emptyset$,

(2) $\mathit{ConsR}\text{-}\mathit{CL} \subset \mathit{Cons}\text{-}\mathit{CL}$.

Proof. The second assertion follows immediately from the first. Let $S_0 = \{a0\tau \mid a \in K, \tau \in \mathcal{R}_{0,1}\}$ and $S_1 = \{\bar{a}1\tau \mid \bar{a} \in \bar{K}, \tau \in \mathcal{R}_{0,1}\}$. Clearly, (S_0, S_1) is classifiable by a CM that waits until seeing the second input and using it as its only conjecture. On the other hand, such a classification procedure cannot be made responsive, as an argument similar to that of Theorem 19 shows. ☒

The proof of Theorem 8 carries over to reveal a hierarchy based on the number of mind changes for consistent classification as well. Many of other results carry over to case of consistent classification. As an example, we show an analogue to Theorem 1 for consistent classification.

Theorem 22. *There are nonempty disjoint sets S_0, S_1 such that*

(1) $S_0 \cup S_1 = \mathcal{R}$,

(2) $(S_0, S_1) \notin$ *Cons-CL*,

(3) $(\rho(S_0), \rho(S_1)) \in$ *Cons-CL*.

Proof. Let S_0, S_1 be defined as in the proof of Theorem 19. Then (1) and (2) follow from that proof.
Finally, (3) will be witnessed by a classification machine knowing whether or not $0 \in K$ and $1 \in K$. More formally, define a CM M as follows.

$$M(f^n) = \begin{cases} 0, & \text{if } f(0) \in K, \\ 1, & \text{if } f(0) \notin K. \end{cases}$$

Then the "finite information" whether or not $0 \in K$ and $1 \in K$ enables the machine M to classify $(\rho(S_0), \rho(S_1))$ consistently. ☒

The proof of Theorem 22 shows that $(\rho(S_0), \rho(S_1))$ is even *finitely* classifiable.

Corollary 23. *There are disjoint sets S_0, S_1 such that*

(1) $S_0 \cup S_1 = \mathcal{R}$,

(2) $(S_0, S_1) \notin$ *Cons-CL*,

(3) $(\rho(S_0), \rho(S_1)) \in$ *FCL*.

8. Classification of Well–known Languages

In this section first we examine the classification of the regular languages (cf. Lewis and Papadimitriou (1981)). The regular languages can be modeled as predicates as well. This is done by fixing an isomorphism between strings over the alphabet of the regular language and the natural numbers. Then a (regular) language represented by a $\{0,1\}$ valued function f is the set of strings that correspond to the natural numbers in the set $\{x \mid f(x) = 1\}$. The details of this encoding will be suppressed as much as possible without sacrificing clarity. By a *positive example* we mean a string that corresponds to a value x such that $f(x) = 1$. Similarly, by a *negative example* we mean a string that corresponds to a value x such that $f(x) = 0$.

Theorem 24. *It is impossible to classify an arbitrary language from positive and negative examples as being either regular or not regular.*

Proof. As a first step, we restate the theorem more formally. Let S_0 be the class of regular languages, i.e., the set of $\{0,1\}$ valued functions that correspond to the regular languages. Let S_1 be all the other languages, e.g. the rest of the $\{0,1\}$ valued functions. We prove that $(S_0, S_1) \notin CL$. Suppose by way of contradiction that M is a CM such that $(S_0, S_1) \in CL(M)$.

We construct a language L over the alphabet $\{a, b\}$, by specifying positive and negative examples. For the sake of presentation we omit the corresponding encoding into a $\{0,1\}$ valued function. Once a string is given as a positive (negative) example, it can never later be given as a negative (positive) example. The *least next example* is the example corresponding to the lexicographically least string that has not yet been presented as either a positive or negative example.

The specification of L is initiated by declaring more and more least next examples as negative until M, given the examples so far, produces a conjectured classification. If M fails to produce a conjecture, then this initialization step will, ultimately, give *every* string as a negative example. In this case, L is the empty language, which is regular. Furthermore, M will fail to correctly classify L.

Let $i \in \{0,1\}$ be the most recent conjecture produced by M after seeing all the examples that have been determined so far. Based on the value of i, new examples are added to L, until (if ever) enough data is added to entice M to produce a new conjecture.

> If $i = 0$: Let x be the least next example. If x corresponds to a string of the form $a^n b^n$, for some n, then add this example as a positive example. Otherwise, the example becomes a negative one.
>
> If $i = 1$: Add the least next example as a negative example.

If the above procedure is successful in always convincing M to change its conjectured classification, then M fails to classify L as either a regular language or not. If M does eventually settle on a classification for L, then one of the two cases above will add the rest of the examples to L. In the first case, when M determines that L is regular, L will end up being a finite variant of the language $\{a^n b^n \mid n \in \mathbb{N}\}$ which is not regular. Since the regular languages are closed under finite variations, L cannot be regular. Hence, M fails to correctly classify L. In the second case, when M determines that L is not regular, L will end up being a finite language. Hence, again, M fails to correctly classify L. ☒

In contrast to Theorem 24, it is possible to separate arbitrarily large subsets of the regular languages from the rest.

Theorem 25. *Let n be an arbitrary natural number. Let S_0 be the set of all regular languages that are recognized by some k-state finite automaton, $k \leq n$. Let S_1 be all the languages not in S_0. Then $(S_0, S_1) \in CL$.*

Proof. There are only finitely many k state finite automata, $k \leq n$, over any given alphabet. A CM that classifies (S_0, S_1) initially conjectures "0" and only changes to "1" when it observes that no automaton in the finite set of n-state automata is consistent with the examples observed as input. ☒

Next we ask whether or not Theorem 25 may be generalized. Disregarding the obvious direction to handle more complex cases of finite sets of languages we consider the problem under what circumstances infinite sets of languages are separable from the rest. As we have already seen, this is not always the case. Moreover, Theorem 24 allows a conclusion that is interesting in its own right. On the one hand, the set of all regular languages is even reliably learnable on the set of all total functions (cf. Blum and Blum (1975)). On the other hand, no algorithm that learns the set of all regular languages can be converted into a classification machine as Theorem 24 shows. Hence, it is interesting to ask for an explanation of that phenomenon.

Analyzing the regular languages from an algorithmic point of view indicates that they possess several favorable properties. First, the set of all regular languages is recursively enumerable. Second, membership in the regular languages is uniformly decidable. Finally, the regular languages possess several structural properties that may be tested algorithmically. Nevertheless, they are not separable, even in the limit, from the rest of all languages. This suggests that their unclassifiability might be caused not by their algorithmic but their topological properties. Namely, from a topological point of view the set of all regular languages is dense, i.e., it consists only of accumulation points. As a consequence, there is no learning algorithm inferring the regular languages that does not exceed any *a priorily* fixed number of allowed mind changes. On the other hand, for any fixed n, the set of all regular languages acceptable by some n-state finite automaton may be obviously inferred within an *a priori* fixed number of mind changes.

Therefore, we will try to generalize Theorem 25 by considering learning algorithms not exceeding a number of mind changes fixed in advance and trying to convert them into classification machines. For that purpose we need some notation. Let Σ be any fixed alphabet. A recursively enumerable set $\mathcal{L} = (L_j)_{j \in \mathbb{N}}$ of languages over Σ is said to be an indexed family if all languages L_j are non–empty and there is an algorithm that uniformly decides membership in L_j for all $j \in \mathbb{N}$ and all strings $s \in \Sigma^*$ (cf. Angluin (1980)). That means $\mathcal{L} = (L_j)_{j \in \mathbb{N}}$ is an indexed family iff there is a recursive function f such that for all numbers j and all strings $s \in \Sigma^*$,

$$f(j,s) = \begin{cases} 1, & \text{if} \quad s \in L_j, \\ 0, & \text{otherwise.} \end{cases}$$

Assuming Σ^* to be lexicographically ordered, we may describe $\mathcal{L}$ by a sequence $\mathcal{F} = (f_j)_{j \in \mathbb{N}}$ of uniformly recursive predicates, where $f_j(s) = f(j,s)$ for all $j \in \mathbb{N}$ and all

$s \in \Sigma^*$. Moreover, in the sequel we identify 2^{Σ^*} with the set of all $\{0,1\}$ valued total functions.

The inferability of indexed families has attracted a lot of attention in learning theory (cf. Zeugmann and Lange (1995) and the references therein). As it turned out, when dealing with the inferability of indexed families the choice of the hypothesis space is of importance. It may effect both the learnability as well as the efficiency (cf. Lange and Zeugmann (1992a), (1993)). In particular, not requiring an IIM to learn within the given enumeration $\mathcal{F} = (f_j)_{j \in \mathbb{N}}$ may reduce the number of necessary mind changes. Therefore, in all what follows we consider *class preserving* learning, i.e., when inferring an indexed family $\mathcal{F} = (f_j)_{j \in \mathbb{N}}$ we allow any suitable chosen hypothesis space $\mathcal{G} = (g_j)_{j \in \mathbb{N}}$ of uniformly recursive predicates such that any predicate in $\mathcal{F}$ possesses a description in $\mathcal{G}$ and any hypothesis g_j describes a language from $\mathcal{F}$, i.e., $range(\mathcal{G}) = range(\mathcal{F})$. Now we are ready to present the next theorem.

Theorem 26. *Let $m \geq 1$, and let $\mathcal{F}$ be any indexed family over some fixed alphabet Σ that can be learned from positive and negative data with at most m mind changes. Then there exists a partition of 2^{Σ^*} into $m+2$ pairwise disjoint classes $\mathcal{F}_0, \ldots, \mathcal{F}_{m+1}$ such that*

(1) $\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_m = \mathcal{F}$,

(2) $(\mathcal{F}_0, \ldots, \mathcal{F}_{m+1}) \in CL$.

Proof. Let $\mathcal{F} \in EX_m(\hat{M})$ with respect to a hypothesis space $\hat{\mathcal{G}}$. It is very tempting to define the wanted partition to be the sets of all $\{0,1\}$ valued functions that are inferred by $\hat{M}$ with exactly i mind changes, for $0 \leq i \leq m$ and $\mathcal{F}_{m+1}$ to be the rest. However, before doing something similar there are some problems we have to overcome. The main difficulty we have to deal with is that we do not know how $\hat{M}$ behaves on inputs f that do not belong to $\mathcal{F}$. In particular, since we want to partition the set of all total $\{0,1\}$ valued function it might even be that $\hat{M}$ is not defined on these inputs. But even if it is, it might converge on functions not belonging to $\mathcal{F}$ after having performed $c \leq m$ mind changes. We overcome these difficulties by applying the characterization theorem of Lange and Zeugmann (1992a), Theorem 12. Applying this theorem we obtain a class preserving hypothesis space $\mathcal{G} = (g_j)$ and a total and consistently working IIM M such that $\mathcal{F} \in EX_m(M)$ with respect to $\mathcal{G}$. That means, on any input f^n, if M outputs a hypothesis j, then $f(x) = g_j(x)$ for all $x \leq n$. Moreover, membership in $\mathcal{G}$ is uniformly decidable. Now we can define the desired partition as follows. For $0 \leq i \leq m$ we set $\mathcal{F}_i = \{ f \mid f$ can be learned by M with exactly i mind changes $\}$, and define $\mathcal{F}_{m+1} = 2^{\Sigma^*} \setminus \cup_{0 \leq i \leq m} \mathcal{F}_i$.

Obviously, $\mathcal{F}_0, \ldots, \mathcal{F}_{m+1}$ is a partition of 2^{Σ^*}. To prove (1) we first recall that any $f \in \mathcal{F}$ is learned by M with at most m mind changes. Hence, $\mathcal{F} \subseteq \mathcal{F}_0 \cup \ldots \cup \mathcal{F}_m$. For the opposite direction suppose by way of contradiction that there is an $f \notin \mathcal{F}$ with $f \in \mathcal{F}_i$ for some $i \in \{0, \ldots, m\}$. Consequently, M, on input $f(0), f(1), \ldots$, produces exactly i mind changes and then it always outputs some hypothesis j. Let $g = g_j$. Taking into account that M works consistently, we may conclude that M

verifies $f(x) = g(x)$ for all $x \in \mathbb{N}$. Hence, $f = g$, and therefore $f \in \mathcal{F}$ since $range(\mathcal{F}) = range(\mathcal{G})$. This contradiction proves (1).

It remains to define a CM $\tilde{M}$ such that $(\mathcal{F}_0, \ldots, \mathcal{F}_{m+1}) \in CL(\tilde{M})$. Let $f \in 2^{\Sigma^*}$ and $n \in \mathbb{N}$. We set

$\tilde{M}(f^n) =$ "Simulate $\hat{M}$ when successively fed $f(0), \ldots, f(n)$. If $\hat{M}$ outputs a hypothesis and has performed exactly i, $0 \leq i \leq m$, mind changes after having read $f(0), \ldots, f(n)$, then output i, and request the next input. Otherwise, output $m+1$, and request the next input."

It is easy to see that $\tilde{M}$ classifies $(\mathcal{F}_0, \ldots, \mathcal{F}_{m+1})$. ☒

The latter theorem directly allows the following corollary.

Corollary 27. *Let $m \geq 1$, and let $\mathcal{F}$ be any indexed family over some fixed alphabet Σ that can be learned from positive and negative data with at most m mind changes. Then it is possible to classify an arbitrary language from positive and negative data as belonging to $\mathcal{F}$ or not.*

Looking at all the results obtained above we see that we have always dealt with complete information concerning the objects to be learned or classified. However, a huge part of language learning theory is devoted to learning from positive data only. Hence, it is only natural to consider classification of languages from positive data. We now briefly examine this issue.

Considering learning or classification from positive data requires some carefulness. First of all, we have to deal with the order of information presentation. Clearly, we cannot assume to receive the data in lexicographical order, since this would implicitly deliver much more information than allowed. Consequently, one demands a CM or IIM to learn on *all* sequences of positive data that eventually contain every string from the language under consideration. More precisely, let L be a language and $t = s_0, s_1, s_2, \ldots$ an infinite sequence of strings from Σ^* such that $range(t) = \{s_k \mid k \in \mathbb{N}\} = L$. Then t is said to be a *positive presentation* for L or, synonymously, a *text*. We define $Text(L)$ to be the set of all texts for L. Moreover, let t be a text and let $x \in \mathbb{N}$: then t_x denotes the initial segment of t of length $x+1$. Note that we do not require a text to be computable. Now we ask whether or not we may extend Theorem 26 to the case of learning from positive data. The answer is twofold. It is still possible to transform a learning algorithm that works with a number of mind changes fixed a priori into a classification machine. Nevertheless, we conjecture that it is no longer possible to partition 2^{Σ^*}. However, we still get a partition of the indexed family. For the sake of readability we present the next theorem in terms of languages.

Theorem 28. *Let $m \in \mathbb{N}$ and let $\mathcal{L}$ be any indexed family over some fixed alphabet that can be learned from positive data with at most m mind changes. Then there exist pairwise disjoint classes $\mathcal{L}_0, \ldots, \mathcal{L}_m$ such that*

(1) $\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_m = \mathcal{L}$,

(2) $(\mathcal{L}_0, \ldots, \mathcal{L}_m) \in CL$.

Proof. Before defining the desired partition we introduce the notion of canonical text that will be helpful in proving the theorem. Let $\mathcal{L}$ be an indexed family and let L any of its languages. Moreover, let $s_0, s_1, \ldots$ be the lexicographically ordered text of Σ^*. The canonical text t of L is obtained as follows. Test sequentially whether $s_z \in L$ for $z = 0, 1, 2, \ldots$ until the first z is found such that $s_z \in L$. Since $L \neq \emptyset$ there must be at least one z fulfilling the test. Set $t_0 = s_z$. We proceed inductively to lengthen this text by appending strings from Σ^* one at a time. We use the notation $t_x \cdot s$ to denote the appending of the string s to t_x, the x^{th} lengthening of the text under construction. For $x \geq 0$ define:

$$t_{x+1} = \begin{cases} t_x \cdot s_{z+x+1}, & \text{if} \quad s_{z+x+1} \in L, \\ t_x \cdot s, & \text{otherwise, where } s \text{ is the last string in } t_x. \end{cases}$$

By assumption, there is an IIM M and a class preserving space of hypothesis $\mathcal{G} = (G_j)_{j \in \mathbb{N}}$ such that M learns any $L \in \mathcal{L}$ from any text $t \in \mathit{Text}(L)$ with at most m mind changes. Now we can define the wanted partition of $\mathcal{L}$ as follows: For $0 \leq i \leq m$ we set $\mathcal{L}_i = \{L \mid L \in \mathcal{L},\ L \text{ can be learned by } M \text{ from its canonical text with exactly } i \text{ mind changes}\}$. Obviously, (1) is fulfilled. It remains to define a CM $\tilde{M}$ that classifies $(\mathcal{L}_0, \ldots, \mathcal{L}_m)$. The main problem we have to deal with is the following. The CM $\tilde{M}$ has to classify every $L \in \mathcal{L}$ from all texts $t \in \mathit{Text}(L)$. Therefore, we cannot simply simulate M and count the number of mind changes, since this number may depend on the particular text the IIM is fed. Hence, our goal must be to simulate M on the canonical text of the language we have to classify. However, in order to generate this canonical text we should know at least one of L's indices in $\mathcal{G}$.

Such an index will be constructed in the limit by the IIM M on any text $t \in \mathit{Text}(L)$. Hence the desired CM $\tilde{M}$ works on arbitrary text $t \in \mathit{Text}(L)$ as follows: It simulates M on t_x, $x \in \mathbb{N}$, until M eventually converges. For any hypothesis j produced by M on t, and hence also for the final and correct hypothesis, $\tilde{M}$ then simulates M on the initial segment t_x^j of the *canonical* text t^j of $L(G_j)$, counts the number of mind changes and outputs the count as its actual hypothesis for classification.

Now we formally define the CM $\tilde{M}$. Let $L \in \mathcal{L}$, $t \in \mathit{Text}(L)$ and $x \in \mathbb{N}$. We set:

$\tilde{M}(t_x) =$ "Compute $j = M(t_x)$.
Compute the number z of mind changes of M when successively fed t_x^j.
If $z \leq m$, then output z and request the next input.
If $z > m$ output nothing and request the next input."

It remains to show that $\tilde{M}$ correctly classifies all the languages from $\mathcal{L}$. Let $L \in \mathcal{L}$, and $0 \leq i \leq m$ be such that L can be learned by M from its canonical text with exactly i mind changes, i.e., $L \in \mathcal{L}_i$. Furthermore, let x be such that both

- $j = M(t_{x+r})$ for all $r \in \mathbb{N}$ (hence $L = L(G_j)$), and

- on t_x^j the machine M performs exactly i mind changes

are satisfied. Obviously, $\tilde{M}(t_y) = i$ for all $y \geq x$. Hence $(\mathcal{L}_0, \ldots, \mathcal{L}_m) \in CL(\tilde{M})$. ☒

9. Conclusions and Open Problems

We have studied the classification of $\{0,1\}$ valued recursive functions, and simultaneously the classification of languages from positive and negative data. Moreover, extending previous work by Wiehagen and Smith (1995) we introduced new models of classification, i.e., classification with a bounded number of mind changes, multi–classification, consistent classification and responsive classification. We related all these classification types to each other, thereby showing what they have in common and where the differences are. However, there are several questions that deserve further study.

First, it would be desirable to gain a deeper understanding under what circumstances the restriction of arbitrary recursive function classes to predicates is classifiable, provided the original function classes are not.

Second, the impact of consistent classification should be investigated in some more detail. Looking at potential applications there are several scenarios where consistent classification is preferable. Nevertheless, as we have seen, this requirement might prevent one from successfully designing a classification machine. Hence, it seems to be highly desirable to elaborate sufficient and necessary conditions for consistent classification. An easy sufficient condition which even might be "practicable" enough for several applications is the following.
Let $S_0, \ldots, S_k \subseteq \mathcal{R}$. We say that the consistency problem of $S_0, \ldots, S_k$ is decidable iff there is a recursive predicate *cons* such that for all $f \in \bigcup_{i \leq k} S_i$, all $n \in \mathbb{N}$ and all "class names" $j \leq k$, *cons* $(j, f^n) = 1$ iff there is $g \in S_j$ such that $g =_n f$ (i. e., *cons* $(j, f^n) = 1$ iff j is a consistent hypothesis of f^n).
Now, if $(S_0, \ldots, S_k) \in CL$ and the consistency problem of $S_0, \ldots, S_k$ is decidable, then $(S_0, \ldots, S_k) \in$ *Cons-CL*.

Third, we would like to find characterizations of all the classification types introduced. Characterizations play an important role in learning theory (cf., e.g., Blum and Blum (1975), Angluin (1980), Zeugmann (1983), Wiehagen (1991), Lange and Zeugmann (1992b)). As it turned out, most of the characterizations obtained lead to a better understanding into the problem how algorithms performing the learning process may be designed. Hence, characterizing classification might yield a deeper insight into the nature of classification. Moreover, this might help to gain a better understanding of the complex relation between classification and learning.

Furthermore, we have mainly dealt with the classification of languages from positive and negative data. Nevertheless, from the point of view of potential applications classification from positive data deserves attention as well.

Finally, a new model of machine discovery has been introduced recently (cf. Mukouchi and Arikawa (1995)). Their model requires the inference device to signal its awareness that no correct hypothesis is in the search space. If it is possible to finitely classify the input as being within the hypothesis space, or outside of it, then learning according the the Mukouchi and Arikawa model is possible. We regard Theorem 28 as a starting point for further research in that direction. In this context, Fulk's (1990) results suggest interesting problems. In particular, Fulk (1990) studied the impact of several demands on the learning power of IIMs, e.g.; prudence, rearrangement independence or set–driveness. Since these requirements reflect postulates of naturalness, it is interesting to consider their influence on the power of CMs.

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