A Note on the Testability of Ramsey's Class

Charles Jordan⋆ and Thomas Zeugmann⋆⋆

Division of Computer Science
Hokkaido University, N-14, W-9, Sapporo 060-0814, Japan
{skip,thomas}@ist.hokudai.ac.jp

Abstract. In property testing, the goal is to distinguish between objects that satisfy some desirable property and objects that are *far* from satisfying it, after examining only a small, random sample of the object in question. Although much of the literature has focused on properties of graphs, very recently several strong results on hypergraphs have appeared. We revisit a logical result obtained by Alon *et al.* [1] in the light of these recent results. The main result is the testability of *all* properties (of relational structures) expressible in sentences of *Ramsey's class*.

Key words: property testing, logic, Ramsey's class

1 Introduction

Alon *et al.* [1] proved the testability of all graph properties expressible in sentences of first-order logic where all quantifier alternations are of the type '$\exists\forall$'. This class is the restriction of Ramsey's [12] class to undirected graphs. Fischer [7] extended this (and other results of [1]) to tournaments. However, Ramsey's [12] class is traditionally not restricted to undirected graphs; any (finite) number of predicate symbols with any (finite) arities may appear. It is therefore natural to ask whether one can extend this result to relational structures.

This result of Alon *et al.* [1] has been influential, and has already been extended several times, cf., e.g., [2, 8, 13]. These extensions have generally focused on an intermediate result: the testability of colorability (and eventually hereditary[1]) properties. In particular, Austin and Tao [4] have recently obtained a strong result: the testability (with one-sided error) of hereditary properties of directed, colored, not necessarily uniform hypergraphs. We return to the logical classification begun by Alon *et al.* [1] and show that this result and generalizations of the remaining parts of the proof in Alon *et al.* [1] combine to give our desired result, the testability of *all* properties expressible in *Ramsey's class*.

2 Preliminaries

Instead of restricting our attention to graphs, we focus on properties of relational structures. We begin by defining vocabularies of such structures.

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[1] A hereditary property is one that is closed under the taking of induced substructures.

Definition 1. *A* vocabulary τ *is a tuple of distinct predicate symbols* $R_i^{a_i}$ *with associated arities* a_i*, i.e.,* $\tau := (R_1^{a_1}, \ldots, R_s^{a_s})$ *.*

The vocabulary (unique up to renaming) of directed graphs is $\tau_G := (E^2)$.

Definition 2. *A* structure A *with vocabulary* τ *is an* $(s+1)$*-tuple,*

$$A := (U, \mathcal{R}_1^A, \ldots, \mathcal{R}_s^A)\,,$$

where U *is a finite universe, and* $\mathcal{R}_i^A$ *is an* a_i*-ary predicate corresponding to the symbol* $R_i^{a_i}$ *of* τ*, i.e.,* $\mathcal{R}_i^A \subseteq U^{a_i}$*.*

The natural numbers are denoted by $\mathbb{N} := \{0, 1, \ldots\}$. For any set U we write $|U|$ to denote the cardinality of U and we generally identify U with the set $\{0, \ldots, |U|-1\}$. The *size* of A is denoted by $\#(A)$ and defined as $\#(A) := |U|$. Let $STRUC^n(\tau)$ be the set of structures with vocabulary τ and size n, and $STRUC(\tau) := \bigcup_{n\in\mathbb{N}} STRUC^n(\tau)$ be the finite structures with vocabulary τ.

A *property* P with vocabulary τ is any subset of $STRUC(\tau)$. We say that a structure A *has* P if $A \in P$. In property testing, we are interested in distinguishing between structures that have some property and those that are *far* from having the property, and so we require a distance measure. Jordan and Zeugmann [9] introduced several possible distances and considered the relationship between the resulting notions of testability. We are proving a positive result, and so it suffices to use only the most restricted variant considered there.

We begin by noting that relations have *subrelations*, for example monadic loops in a binary predicate. In property testing, it can be useful and is more restrictive (see Jordan and Zeugmann [9]) to consider these subrelations as separate relations when defining the distance between structures. We first define the syntactic notion of *subtype* before proceeding to subrelations.

Definition 3. *A* subtype S *of a predicate symbol* $R_i^{a_i}$ *is any partition of the set* $\{1, \ldots, a_i\}$*.*

For example, graphs have a single, binary predicate symbol E^2 which has two subtypes: $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$, corresponding to loops and non-loops respectively. Let $SUB(R_i^{a_i})$ denote the set of subtypes of predicate symbol $R_i^{a_i}$.

Definition 4. *Let* $A \in STRUC(\tau)$ *be a structure with vocabulary* τ *and universe* U*, and let* S *be a subtype of predicate symbol* $R_i^{a_i} \in \tau$*. We define* $s^U(S)$*, the* tuples that belong to S*, as the set* $(x_1, \ldots, x_{a_i}) \in U^{a_i}$ *satisfying the following condition. For every* $1 \le j, k \le a_i$*,* $x_j = x_k$ *iff* j *and* k *are contained in the same element of* S*. The* subrelation $s^A(S)$ *of* A *corresponding to* S *is* $s^A(S) := s^U(S) \cap \mathcal{R}_i^A$*.*

Returning to our example of graphs, the sets of loops and non-loops are the subrelations of $\mathcal{E}$ corresponding to the subtypes $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$ of E^2, respectively. We denote the symmetric difference of sets U and V by $U \triangle V$,

$$U \triangle V := (U \backslash V) \cup (V \backslash U)\,.$$

Definition 5. *Let $A, B \in STRUC^n(\tau)$ be structures with vocabulary τ and size n. The* distance *between A and B is*

$$\mathrm{mrdist}(A,B) := \max_{R_i^{a_i} \in \tau} \max_{S \in SUB(R_i^{a_i})} \frac{|s^A(S) \triangle s^B(S)|}{n^{|S|}}.$$

The distance between structures is the fraction of assignments that differ in the most different subtype. The distance between structures generalizes to distance from properties in the usual way.

Definition 6. *Let $A \in STRUC^n(\tau)$ be a structure with vocabulary τ and size n, and let $P \subseteq STRUC(\tau)$ be a property with vocabulary τ. The* distance *between A and P is*

$$\mathrm{mrdist}(A,P) := \min_{B \in P \cap STRUC^n(\tau)} \mathrm{mrdist}(A,B).$$

If $P \cap STRUC^n(\tau)$ is empty, we let the distance be infinite.

We are now able to define property testing itself.

Definition 7. *Let $P \subseteq STRUC(\tau)$ be a property with vocabulary τ. An (ε, q)-*tester *T for P is a probabilistic algorithm which satisfies the following conditions, when given ε and access to an oracle that answers queries for the universe size $\#(A)$ and for the assignment of any tuple $(x_1, \ldots, x_{a_i}) \in \mathcal{R}_i^A$:*

1. *If $A \in P$, then T accepts with probability at least $2/3$.*
2. *If $\mathrm{mrdist}(A,P) \geq \varepsilon$, then T rejects with probability at least $2/3$.*
3. *T makes at most q queries.*

In property testing, we are particularly interested in properties that can be tested with a number of queries depending only on ε.

Definition 8. *A property P is* testable *if there exists a function $c(\varepsilon)$ such that for every $\varepsilon > 0$, there exists an $(\varepsilon, c(\varepsilon))$-tester for P.*

This is a non-uniform definition because we do not require the testers to be, e.g., computable given ε. There exist properties that are non-uniformly testable but not uniformly testable (see, e.g., Alon and Shapira [3]). Our results hold in either case[2] (i.e., one can replace all occurrences of "testable" below by "uniformly testable" and maintain correctness) and so we will not distinguish between them.

We also require logical definitions. These definitions are standard and we review them quickly. See Enderton [6] for an introduction to logic and Börger *et al.* [5] for background on classification problems.

The first-order language of τ is the closure of the atomic formulae $x_i = x_j$ and $R_i^{a_i}(x_1, \ldots, x_{a_i})$ for variable symbols x_k under Boolean connectives $\wedge$, $\vee$ and $\neg$ and first-order quantifiers $\forall$ and $\exists$. We do not allow ordering or arithmetic. These sentences are interpreted in the usual way and so, for a structure $A \in STRUC(\tau)$

[2] In the uniform case, we must restrict Lemma 1 to decidable properties. All properties considered here are clearly decidable.

and sentence φ of vocabulary τ, we can decide whether $A \models \varphi$. Logical sentences define properties; if φ is a sentence with vocabulary τ, then it defines property $P_\varphi := \{A \mid A \in STRUC(\tau), A \models \varphi\}$.

Our logic does not contain arithmetic or ordering, and so all properties expressible in it are closed under isomorphisms. We formalize this as follows.

Definition 9. *Let $A, B \in STRUC^n(\tau)$ be two structures with universe U and vocabulary τ. We say that A and B are* isomorphic *if there exists a bijection $f\colon U \to U$ such that for all $1 \le i \le s$ and $x_1, \ldots, x_{a_i} \in U$, we have*

$$(x_1, \ldots, x_{a_i}) \in \mathcal{R}_i^A \iff (f(x_1), \ldots, f(x_{a_i})) \in \mathcal{R}_i^B .$$

Definition 10. *Let P be a property with vocabulary τ. We say that P is* closed under isomorphisms *if for all isomorphic A, B, A has P iff B has P.*

Our goal is a classification of the syntactic subclasses of first-order logic according to their testability. These subclasses are traditionally formulated as prefix-vocabulary fragments. Here we are only interested in Ramsey's class, and so we omit more general definitions, see, e.g., Börger *et al.* [5] for details.

Ramsey's class is denoted $[\exists^*\forall^*, all]_=$. This is the set of sentences of first-order predicate logic in prenex normal form, where all existential quantifiers precede all universal quantifiers. Function symbols do not appear, but any number of predicate symbols of any arity may appear, as may the special atomic predicate =. Ramsey's class has a number of nice algorithmic properties. For example, Ramsey [12] showed the satisfiability problem for this class is decidable, and Lewis [11] showed it to be NEXPTIME-complete. Kolaitis and Vardi [10] proved a 0-1 law holds for existential *second*-order sentences, if the first-order part is in Ramsey's class. The class that Alon *et al.* [1] proved testable is essentially the restriction of Ramsey's class to graphs, denoted $[\exists^*\forall^*, (0, 1)]_=$.

3 Testability of Ramsey's Class

We show that all properties expressible in $[\exists^*\forall^*, all]_=$ are testable. The proof follows that of Alon *et al.* [1]. First, we show that their notion of *indistinguishability* preserves testability after generalizing to relational structures. Then, we prove that all sentences in our class define properties that are indistinguishable from instances of a generalized colorability problem. Finally, we show that all such problems are hereditary and therefore testable in the setting defined by Austin and Tao [4]. This implies testability under our definitions, giving the following.

Theorem 1. *All sentences in $[\exists^*\forall^*, all]_=$ are testable.*

Alon *et al.* [1] introduced the concept of indistinguishability and showed that it preserves testability of graph properties. We begin by extending their notion to relational properties.

Definition 11. *Let $P_1, P_2 \subseteq STRUC(\tau)$ be properties with vocabulary τ that are closed under isomorphisms. We say that P_1 and P_2 are* indistinguishable *if for every $\varepsilon > 0$ there exists an $N := N(\varepsilon) \in \mathbb{N}$ such that the following holds for all $n > N$. For every $A \in STRUC^n(\tau)$, if A has property P_1, then* $\mathrm{mrdist}(A, P_2) < \varepsilon$ *and if A has P_2, then* $\mathrm{mrdist}(A, P_1) < \varepsilon$.

The importance of indistinguishability is that it preserves testability.

Lemma 1. *Let $P_1, P_2 \subseteq STRUC(\tau)$ be indistinguishable properties with vocabulary τ. Property P_1 is testable iff P_2 is testable.*

The proof of Lemma 1 is a simple extension of the proof by Alon *et al.* [1] and is omitted due to space constraints. Next, we will show that all sentences in $[\exists^*\forall^*, all]_=$ define properties that are indistinguishable from instances of a generalized colorability problem. We begin by defining the colorability problem.

For any fixed set F of structures with vocabulary τ, some positive number of colors c, and functions that assign a color between 1 and c to each element of each structure in F, we define the F-colorability problem as follows. A structure $A \in STRUC(\tau)$ is F-colorable if there exists some (not necessarily proper) c-coloring of A such that A does not contain any induced substructures isomorphic to a member of F. We let P_F be the set of structures that are F-colorable.

For example, we can consider the case of graphs and let F contain c copies of K_2. We enumerate these copies in some fashion from 1 to c, and for copy i, color both vertices with i. The resulting problem is of course the usual (k- or equivalently) c-colorability. The following is a straightforward generalization of the proof by Alon *et al.* [1].

Lemma 2. *Let φ be any first-order sentence in the class $[\exists^*\forall^*, all]_=$. There exists an instance of the F-colorability problem that is indistinguishable from P_φ, the property defined by φ.*

Proof. Let $\varepsilon > 0$ be arbitrary and $\varphi := \exists x_1 \ldots \exists x_t \forall y_1 \ldots \forall y_u : \psi$ be any first-order formula with quantifier-free ψ and vocabulary τ. We note, as did Alon *et al.* [1], that we can restrict our attention to formulas ψ where it is sufficient to consider only cases where the variables are bound to distinct elements. This is because, given any ψ', we can construct a ψ satisfying this restriction that is equivalent on structures with at least $t + u$ elements, and the smaller structures do not matter in the context of indistinguishability.

Let $P = \{A \mid A \in STRUC(\tau), A \models \varphi\}$ be the property defined by φ. We now define an instance of F-colorability that we will show to be indistinguishable from P. We denote our c colors by the elements of

$$\{(0,0)\} \cup \{(a,b) \mid 1 \le a \le \pi_1, 1 \le b \le \pi_2, a, b \in \mathbb{N}\}\,.$$

Here, π_1 is the number of distinct structures of vocabulary τ with exactly t elements, $\pi_1 := 2^{\sum_{1\le i\le s} t^{a_i}}$. Similarly, we denote by π_2 the number of ways it is possible to "connect" or "add" a single element to some existing, fixed t-element structure of vocabulary τ, i.e., $\pi_2 := 2^{\sum_{1\le i\le s}\sum_{1\le j\le a_i-1}\binom{a_i}{j}t^{a_i-j}}$. We will use fixed enumerations of these π_1 structures with t elements and π_2 ways of connecting an additional element to a fixed t element structure.

We impose on the coloring of the structure the following restrictions. Each can be expressed by prohibiting finite sets of colored induced substructures.

(1) The color $(0,0)$ may be used at most t times. Therefore, we prohibit all $(t+1)$-element structures that are colored completely with $(0,0)$.
(2) The graph must be colored using only $\{(0,0)\} \cup \{(a,b) \mid 1 \leq b \leq \pi_2\}$ for some fixed $a \in \{1,\ldots,\pi_1\}$. Therefore, we prohibit all two-element structures colored $((a,b),(a',b'))$ with $a \neq a'$.
(3) We now consider some fixed coloring of a u-element structure V, whose universe we identify with $\{v_1,\ldots,v_u\}$. We assume that this coloring satisfies the previous restriction and that color $(0,0)$ does not appear. We must decide whether to prohibit this structure. In order to do so, we first take the fixed a guaranteed by the previous restriction, and consider the t-element structure E, whose universe we identify with $\{e_1,\ldots,e_t\}$, that is the a^{th} structure in our enumeration of t element structures. We connect each v_i to E in the following way. If v_i is colored (a,b), we use the b^{th} way of connecting an additional element to a t-element structure in our enumeration. We denote the resulting $(t+u)$-element structure as M and allow (do not prohibit) M iff M is a model of ψ when we replace x_i with u_i and y_j with v_j.

We now show that the resulting F-colorability problem is indistinguishable from P. Assume that we are given an $A \models \varphi$. Color the t vertices existentially bound to the x_i with $(0,0)$. Then, we can color all remaining vertices v_i with (a,b), where a corresponds to the substructure induced by $\{x_1,\ldots,x_t\}$ in our enumeration of t-element structures, and b corresponds to the connection between v_i and $\{x_1,\ldots,x_t\}$. It is easy to see that this coloring satisfies the restrictions of our F-colorability problem. We have not made any modifications to the structure and so $\mathrm{mrdist}(A,P_F)=0$.

Next, we assume that we are given a structure with a coloring that satisfies our restrictions. We will show that we can obtain a model of φ by making only a small number of modifications. First, if there are less than t elements colored $(0,0)$, we arbitrarily choose additional elements to color $(0,0)$ so that there are exactly t such elements. We will denote these t elements with $\{e_1,\ldots,e_t\}$. Restriction (2) guarantees that all colors which are not $(0,0)$ share the same first component. Let a be this shared component. We make the structure induced by $\{e_1,\ldots,e_t\}$ identical to the a^{th} structure in our enumeration of t-element structures, requiring at most $\sum_{1\leq i\leq s} t^{a_i} = O(1)$ modifications. Next, for each element v_i that is colored (a,b) with $a,b \neq 0$, we modify the connections between v_i and $\{e_1,\ldots,e_t\}$ in order to make these connections identical to the b^{th} way of making such connections in our enumeration. This requires at most

$$(n-t)\sum_{1\leq i\leq r}\ \sum_{1\leq j\leq a_i-1}\left[\binom{a_i}{j} t^{a_i-j}\right] = O(n)$$

additional modifications, all of which are to non-monadic subrelations. Binding x_i to e_i, the resulting structure is a model of φ. We made at most $O(1)$ modifications to monadic subrelations and $O(n)$ modifications to non-monadic subrelations, and so $\operatorname{mrdist}(A,P) \leq \max\{O(1)/n, O(n)/\Omega(n^2)\} = o(1) < \varepsilon$, where the inequality holds for sufficiently large n.

Therefore, all such properties P are indistinguishable from instances of F-colorability. $\square$

A *hereditary property* of relational structures is one which is closed under taking induced substructures. F-colorability is clearly a hereditary property; if A is F-colorable, then so are its induced substructures. However, the definitions of Austin and Tao [4] are significantly different from ours and so we explicitly reduce the following translation in our setting to their result.

Theorem 2 (Translation of Austin and Tao [4]). *Let P be a hereditary property of relational structures which is closed under isomorphisms. Then, property P is testable with one-sided error.*

Before reducing Theorem 2 to its statement in [4], we first briefly introduce their definitions. All of the definitions in Subsection 3.1 are from Austin and Tao [4], although we omit definitions which are not necessary for our purposes.

3.1 Framework of Austin and Tao [4]

We begin by introducing their analogue of vocabularies: *finite palettes.*

Definition 12. *A* finite palette K *is a sequence $K := (K_j)_{j=0}^{\infty}$ of finite sets, of which all but finitely-many are singletons. The singletons are called* points *and denoted* pt. *A point is called* trailing *if it occurs after all non-points.*

We will write $K = (K_0, \dots, K_k)$, omitting trailing points and call k the *order* of K. We use the elements of K_j to *color* the j-ary edges in hypergraphs.

Definition 13. *A* vertex set V *is any set which is at most countable. If V, W are vertex sets, then a* morphism f *from W to V is any injective map $f\colon W \to V$ and the set of such morphisms is denoted* $\operatorname{Inj}(W,V)$. *For $N \in \mathbb{N}$, we denote the set $\{1,\dots,N\}$ by $[N]$.*

Of course, $[N]$ is a vertex set. Our structures are finite so we are mostly interested in *finite* vertex sets. Next, we define the analogue of relational structures.

Definition 14. *Let V be a vertex set and K be a finite palette. A K-*colored hypergraph G *on V is a sequence $G := (G)_{j=0}^{\infty}$, where each $G_j\colon \operatorname{Inj}([j],V) \to K_j$ is a function. Let $K^{(V)}$ be the set of K-colored hypergraphs on V.*

Only finitely many of the K_j are not points, and so only finitely many G_j are non-trivial. The G_j assign colors from K_j to the morphisms in $\operatorname{Inj}([j],V)$. In our relational setting, this set of morphisms corresponds to the set of j-ary tuples $(x_1,\dots,x_j)$ with pairwise distinct components.

Before defining hereditary K-properties, we need one last technical definition.

Definition 15. *Let V, W be vertex sets and $f \in \mathrm{Inj}(W, V)$ be a morphism from W to V. The* pullback map $K^{(f)} \colon K^{(V)} \to K^{(W)}$ *is*

$$\left(K^{(f)}(G)\right)_j (g) := G_j(f \circ g)\,,$$

for all $G = (G_j)_{j=0}^{\infty} \in K^{(V)}$, $j \geq 0$ and $g \in \mathrm{Inj}([j], W))$. If $W \subseteq V$ and $f \in \mathrm{Inj}(W, V)$ is the identity map on W, we abbreviate

$$G \downharpoonleft_W := K^{(f)}\,.$$

Abusing notation, the pullback map $K^{(f)}$ maps K-colored hypergraphs on V to those on W, by assigning the color of $f \circ g$ to g, for all tuples g. Note that $G \downharpoonleft_W$ is equivalent to the induced subhypergraph on W. For notational clarity, we reserve P for properties of relational structures and use $\mathcal{P}$ to denote properties of hypergraphs.

Definition 16. *Let $K = (K_j)_{j=0}^{\infty}$ be a finite palette. A* hereditary K-property *$\mathcal{P}$ is an assignment $\mathcal{P} \colon V \mapsto \mathcal{P}^{(V)}$ of a collection $\mathcal{P}^{(V)} \subseteq K^{(V)}$ of K-colored hypergraphs for every finite vertex set V such that*

$$K^{(f)}(\mathcal{P}^{(V)}) \subseteq \mathcal{P}^{(W)}$$

for every morphism $f \in \mathrm{Inj}(W, V)$ between finite vertex sets.

Finally, we state the definition of (one-sided error) testability used by Austin and Tao [4]. Here, for a vertex set V and $c \in \mathbb{N}$, we write $\binom{V}{c} := \{V' \mid V' \subseteq V, |V'| = c\}$ to denote the set of subsets of V with exactly c elements.

Definition 17. *Let K be a finite palette with order $k \geq 0$ and $\mathcal{P}$ be a hereditary K-property. Property $\mathcal{P}$ is* testable with one-sided error *if for every $\varepsilon > 0$, there exists $N \geq 1$ and $\delta > 0$ satisfying the following. For all vertex sets V with $|V| \geq N$, if $G \in K^{(V)}$ satisfies*

$$\frac{1}{\left|\binom{V}{N}\right|} \left| \left\{ W \mid W \in \binom{V}{N}, G \downharpoonleft_W \in \mathcal{P}^{(W)} \right\} \right| \geq 1 - \delta\,, \tag{1}$$

then there exists a $G' \in \mathcal{P}^{(V)}$ satisfying

$$\frac{1}{\left|\binom{V}{k}\right|} \left| \left\{ W \mid W \in \binom{V}{k}, G \downharpoonleft_W \neq G' \downharpoonleft_W \right\} \right| \leq \varepsilon\,. \tag{2}$$

To see that this is a variant of testability, it is easiest to consider the contrapositive. If there is a G' satisfying (2), then G is not ε-far from $\mathcal{P}$, using the implicit distance measure based on the fraction of differing induced subhypergraphs of size k. If there is no such G' (i.e., G is ε-far from $\mathcal{P}$) and $\mathcal{P}$ is testable, then (1) must not hold. That is, there are *many* induced subhypergraphs of size N that do not have $\mathcal{P}$. The definition is for hereditary $\mathcal{P}$, and so if G has $\mathcal{P}$, then so do all induced subhypergraphs. This allows the construction of testers.

Finally, we can state one of the main results of Austin and Tao [4].

Theorem 3 (Austin and Tao [4]). *Let K be a finite palette and let $\mathcal{P}$ be a hereditary K-property. Then, $\mathcal{P}$ is testable with one-sided error.*

In the following subsection we will map our vocabularies, structures and properties to this setting. We will then show that hereditary properties in our setting correspond to hereditary properties (in the sense of Definition 16) here, and that testability in the sense of Definition 17 implies testability of the original relational properties. That is, we explicitly reduce Theorem 2 to Theorem 3.

3.2 Reducing Theorem 2 to Theorem 3

We begin by mapping vocabulary $\tau = \{R_1^{a_1}, \ldots, R_s^{a_s}\}$ to a finite palette $K_\tau = (K_i)_{i=0}^{\infty}$. We use the color of a "tuple" to represent the set of assignments on it. The difference between the set of j-ary tuples over a finite universe U and $\mathrm{Inj}([j], U)$ is that the latter does not permit repeated components. If $S \in SUB(R_i^{a_i})$ has $|S| < a_i$, then the corresponding subrelation consists of tuples with repeated components. We treat such S as relations with arity $|S|$ and no repeated components. Here, $\mathfrak{S}(n,k)$ is the Stirling number of the second kind.

For $a \geq 1$, let $P_a := \{R_i^{a_i} \mid R_i^{a_i} \in \tau, a_i = a\}$ be the set of predicate symbols with arity a. We now define palette K. Let $K_0 := \mathrm{pt}$ and $K_i := \left[2^{\sum_{j \geq i} |P_j| \mathfrak{S}(j,i)}\right]$. There are finitely-many predicate symbols and so only finitely-many $K_i \neq \mathrm{pt}$.

Let $S_a := \{S_a^i \mid S_a^i \in SUB(R_i^{a_i}), |S_a^i| = a, 1 \leq i \leq s\}$ be the set of subtypes with cardinality a for all $a \geq 1$. Now, $2^{|S_a|} = |K_a|$ and we have exactly enough colors to encode the set of assignments of the a-ary subtypes on a-ary tuples.

We will now define a map h from relational structures A on universe U to hypergraphs $G_A \in K^{(U)}$. For any $S_a^i \in S_a$, there is a bijection

$$r(S_a^i) \colon s^U(S_a^i) \to \{(x_1, \ldots, x_a) \mid x_i \in U, x_i \neq x_j \text{ for } i \neq j\}$$

from $s^U(S_a^i)$ to the a-ary tuples without duplicate components, formed by removing the duplicate components. That is, $r(S_a^i)$ maps $(x_1, \ldots, x_{a_i})$ to $(x_{i_1}, \ldots, x_{i_a})$ where $1 \leq i_1 < i_2 < \ldots < i_a \leq a_i$. We can now define $G_A = h(A)$.

For $j > 0$, we define $G_j \colon \mathrm{Inj}([j], U) \to K_j$ as follows. Assign to $f \in \mathrm{Inj}([j], U)$ the color encoding the set of assignments of the subtypes S_j on $(f(1), \ldots, f(j))$, using the inverses $(r(S_j^i))^{-1}$ to get assignments for subtypes of high-arity relations. For $j = 0$, $\mathrm{Inj}([j], U) = \emptyset$ and $K_0 = \mathrm{pt}$ and we can use a trivial map.

Of course, we extend the map to properties in the obvious way. If P is a property of relational structures, we let $\mathcal{P}^{(U)} := \{h(A) \mid A \in P\}$. Formally, we define $\mathcal{P}(U) := \mathcal{P}^{(U)}$, but there is a small technical point. We have identified finite universes with subsets of the naturals, allowing us to call $STRUC(\tau)$ a *set*. However, Definition 13 allows a vertex set to be *any* finite set and Definition 16 requires hereditary hypergraph properties to be closed under bijections between vertex sets. To remedy this, for each finite vertex set W, we fix a[3] bijection

[3] Our properties are closed under isomorphisms, so any fixed bijection is acceptable.

$g^W : W \to \{0, \ldots, |W| - 1\}$. We then define $\mathcal{P} := h(P)$ formally as

$$\mathcal{P}(W) := \begin{cases} \mathcal{P}^{(W)}, & \text{if } W = \{0, 1, \ldots, |W| - 1\}; \\ K^{\left(g^W\right)}\left(\mathcal{P}^{(\{0,\ldots,|W|-1\})}\right), & \text{otherwise.} \end{cases}$$

Hereditary relational properties are mapped to hereditary hypergraph properties, which are testable in the sense of Definition 17 by Theorem 3.

Lemma 3. *If P is a hereditary property of relational structures, then $h(P)$ is a hereditary property of hypergraphs.*

Proof. Let P be a hereditary property of relational structures with vocabulary τ. Assume that $\mathcal{P} := h(P)$ does not satisfy Definition 16. Then, there exist finite vertex sets V and W, and a morphism $f' \in \mathrm{Inj}(W, V)$ such that

$$K^{(f)}(\mathcal{P}^{(V)}) \nsubseteq \mathcal{P}^{(W)}. \tag{3}$$

Since f' exists, $\mathrm{Inj}(W, V)$ cannot be the empty set and so $|V| \geq |W|$. Let $U_V := \{0, \ldots, |V|-1\}$ and $U_W := \{0, \ldots, |W|-1\}$. By the definition of $\mathcal{P}$, we can fix bijections $g^V : V \to U_V$ and $g^W : W \to U_W$ such that $\mathcal{P}^{(V)} = K^{\left(g^V\right)}\left(\mathcal{P}^{(U_V)}\right)$ and $\mathcal{P}^{(W)} = K^{\left(g^W\right)}\left(\mathcal{P}^{(U_W)}\right)$. By the definition of $\mathcal{P} = h(P)$, this implies

$$K^{(f)}\left(K^{\left(g^V\right)}\left(\mathcal{P}^{(U_V)}\right)\right) \nsubseteq K^{\left(g^W\right)}\left(\mathcal{P}^{(U_W)}\right).$$

Bijections are invertible, and so this implies

$$K^{\left(g^V \circ f \circ \left(g^W\right)^{-1}\right)}\left(\mathcal{P}^{(U_V)}\right) \nsubseteq \mathcal{P}^{(U_W)}.$$

Rename $f' := g^V \circ f \circ \left(g^W\right)^{-1}$ and note $f' \in \mathrm{Inj}(U_W, U_V)$. Let $A' \in \mathcal{P}^{(U_V)}$ be such that $K^{(f')}(A') \notin \mathcal{P}^{(U_W)}$.

We defined $\mathcal{P}$ as $h(P)$ for a hereditary property P of relational structures. Property P is closed under isomorphisms, and so there is an $A := h^{-1}(A') \in P \cap STRUC^{|U_V|}(\tau)$ such that the $|U_W|$-element substructure induced by $\{a \mid a = f'(u) \text{ for some } u \in U_w\}$ does not have P. This contradicts the hereditariness of P and so $\mathcal{P}$ must be hereditary in the sense of Definition 16. □

We mapped hereditary relational properties to hereditary hypergraph properties, which are testable by Theorem 3. We will show this implies testability of the original properties.

Definition 18. *Let $A, B \in STRUC^n(\tau)$ be structures with vocabulary τ and universe $U := \{0, \ldots, n-1\}$ of size n, $k := \max_i a_i$ be the maximum arity of the predicate symbols, and $h\colon STRUC^n(\tau) \to K^{(U)}$ be the map defined above. The* h-distance *between A and B is*

$$\mathrm{hdist}(A, B) := \frac{1}{\left|\binom{U}{k}\right|} \left|\left\{W \mid W \in \binom{U}{k}, h(A) \downharpoonright_W \neq h(B) \downharpoonright_W\right\}\right|.$$

We now relate the two distances with the following simple lemma.

Lemma 4. *Let $A, B \in STRUC^n(\tau)$ be relational structures with vocabulary τ and size n. Then,* $\mathrm{hdist}(A, B) \geq \mathrm{mrdist}(A, B)$.

Proof. Assume that $\mathrm{mrdist}(A, B) = \varepsilon$. Then, there exists a predicate symbol $R_i^{a_i} \in \tau$ and subtype $S \in SUB(R_i^{a_i})$ such that $\left|s^A(S) \triangle s^B(S)\right| / n^{|S|} = \varepsilon$. Let $k := \max_i a_i$ and let the universe of both structures be $U_n := \{0, \ldots, n-1\}$.

Consider a random permutation of the universe (i.e., a bijection $r\colon U_n \to U_n$) chosen uniformly from the set of such permutations. The probability that the substructures induced on $\{r(0), \ldots, r(k-1)\}$ differ in A and B is $\mathrm{hdist}(A, B)$. The probability that the first $|S|$ elements, i.e. $\{r(0), \ldots, r(|S|-1)\}$, differ in $s^A(S)$ and $s^B(S)$ is ε and so $\mathrm{hdist}(A, B) \geq \varepsilon$. □

Equality is obtained when $|S| = k$. It is possible to show that the two distances differ by at most a constant factor, and so the corresponding notions of testability are essentially equivalent. However, Lemma 4 suffices for our purposes.

Lemma 5. *Let $P \subseteq STRUC(\tau)$ be a property of relational structures which is mapped by h to a property of hypergraphs that is testable with one-sided error. Then, P is testable with one-sided error.*

Proof. Let $\mathcal{P} := h(P)$ be the hypergraph property which P is mapped to. We will show that the following is an ε-tester for P with one-sided error. Let $N \geq 1$, $\delta > 0$ be the constants of Definition 17 for ε. Assume that we are testing a structure $A \in STRUC^n(\tau)$ and recall that $U = \{0, \ldots, n-1\}$.

1. If $\#(A) \leq N$, query the entire structure and decide exactly whether $A \in P$.
2. Otherwise, repeat the following $q(\delta)$ times.
   (a) Uniformly select N elements and query the induced substructure.
   (b) If it has P, continue. Otherwise, reject.
3. Accept if all of the induced substructures had P.

If $A \in P$, then all induced substructures have P because P is hereditary and the tester accepts with probability 1. Next, assume $\mathrm{mrdist}(A, P) > \varepsilon$. We use Definition 17 to show the tester will find a witness for $A \notin P$ with probability at least 2/3. By Lemma 4, $\mathrm{hdist}(A, P) \geq \mathrm{mrdist}(A, P) > \varepsilon$. We assumed $h(P)$ is hereditary, and so (by Theorem 3) it is testable in the sense of Definition 17. The probability that a uniformly chosen N-element substructure does not have P is at least δ. We use $q(\delta)$ to amplify the success probability from δ to 2/3. □

This completes the proof of Theorem 1. All properties expressible in Ramsey's class are indistinguishable from instances of F-colorability. Indistinguishability preserves testability and so it sufficed to show that these instances are testable. All instances of F-colorability are hereditary relational properties, which are testable by Theorem 2, which we reduced to the statement by Austin and Tao [4].

4 Conclusion

We have revisited the positive result obtained by Alon *et al.* [1] in the light of a strong result obtained recently by Austin and Tao [4]. We have shown that this allows us to extend the proof that properties expressible in Ramsey's class are testable, from undirected, loop-free graphs to arbitrary relational structures.

A more direct proof for the testability of Ramsey's class would be interesting, especially if it results in better query complexity. It would also be interesting to consider the testability of additional prefix classes and connections with other classifications (such as, e.g., that for the finite model property).

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