A Short Proof of the Convergence of IFS

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In this note, a short proof is given that Iterated Function Systems (IFS) always converge, i.e. the *attractor* always exists. The reader should be familiar with some basic results from analysis and topology.

Definition. Let $\mathcal{X} = \{A \subset \mathbb{R}^2 : \ A \text{ is compact}\}$ be the space of all compact subsets of the Euclidean plane.

Definition. Let $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be the mapping

$$d(A, B) = \max \big\{ \max_{a \in A} \min_{b \in B} \|b - a\|, \max_{b \in B} \min_{a \in A} \|b - a\| \big\},$$

where $\|x\| = \sqrt{x_1^2 + x_2^2}$ denotes the Euclidean norm (square norm). The maxima and minima are attained since A and B are compact.

Proposition. *The function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defines a metric on $\mathcal{X}$.*

Proof. Symmetry is straightforward. The implication $d(A, B) = 0 \Rightarrow A = B$ follows from compactness of A and B. Finally,

$$\begin{aligned} \max_{a \in A} \min_{c \in C} |c - a| &\leq \max_{a \in A} \min_{b \in B} \min_{c \in C} \{|c - b| + |b - a|\} \\ &\leq \max_{a \in A} \min_{c \in C} |c - b| + \max_{b \in B} \min_{c \in C} |b - a| \leq d(A, B) + d(B, C) \end{aligned}$$

implies the triangle inequality. □

Proposition. *$(\mathcal{X}, d)$ is a complete metric space.*

Proof. Let $(A_n)_{n \geq 1}$ be a Cauchy sequence in $\mathcal{X}$. Then $\varepsilon_n = \sup_{m \geq n} d(A_m, A_n)$ tends to 0 as $n \to \infty$. For $A \in \mathcal{X}$ and $\varepsilon \geq 0$, let

$$[A]_\varepsilon = \{a + x : a \in A, \, x \in \mathbb{R}^2, \, \|x\| \leq \varepsilon\}.$$

Then $[A]_\varepsilon \in \mathcal{X}$ holds. Define the candidate for $\lim A_n$ as

$$A_\infty = \bigcap_{n=1}^{\infty} [A_n]_{\varepsilon_n}.$$

For each n, we have obviously $d([A_n]_{\varepsilon_n}, A_n) \leq \varepsilon_n$. Then $A_n \to A_\infty$ follows from $d(A_\infty, A_n) \leq d(A_\infty, [A_n]_{\varepsilon_n}) + d([A_n]_{\varepsilon_n}, A_n) \leq 3\varepsilon_n$, which holds due to the

Claim: $d(A_\infty, [A_n]_{\varepsilon_n}) \leq 2\varepsilon_n$.

In order to verify the claim, observe that $d(A_\infty, [A_n]_{\varepsilon_n}) = \max_{x \in [A_n]_{\varepsilon_n}} d(x, A_\infty)$ because of $A_\infty \subset [A_n]_{\varepsilon_n}$. For each $x \in [A_n]_{\varepsilon_n}$ and $m \geq n$, there is $a \in A_n$ and $y_m \in A_m$ such that $\|x - a\| \leq \varepsilon_n$ and $\|a - y_m\| \leq \varepsilon_n$, hence $\|x - y_m\| \leq 2\varepsilon_n$.

The sequence y_m has an accumulation point y, which is contained in all $[A_k]_{\varepsilon_k}$ for $k \geq 1$, since $y_m \in A_m \subset [A_k]_{\varepsilon_k}$ holds for $m \geq k \geq 1$. This implies $y \in A_\infty$. From $\|x - y_m\| \leq 2\varepsilon_n$, we conclude $\|x - y\| \leq 2\varepsilon_n$. This establishes the claim and completes the proof. □

Definition. A (linear) IFS is a function $\Phi : \mathcal{X} \to \mathcal{X}$ which consists of finitely many affine transformations:

$$\Phi(A) = \bigcup_{i=1}^{n} \{U_i x + v_i : x \in A\},$$

where $U_1, \ldots, U_n \in \mathbb{R}^{2\times 2}$ are linear maps having norm strictly less than 1, $\|U_i\| = \max\{\|U_i x\| : \|x\| \leq 1\} < 1$, and $v_1, \ldots, v_n \in \mathbb{R}^2$ are vectors.

Lemma. *An IFS Φ is a contraction on $\mathcal{X}$ with respect to d.*

Proof. Let $q = \max_i \|U_i\| < 1$ and $A, B \in \mathcal{X}$, then $d(A, B) = \|a - b\|$ for suitable $a \in A$ and $b \in B$. Now $d\big(\Phi(A), \Phi(B)\big) \leq q\|a - b\|$ is straightforward. □

Theorem. *Each IFS Φ has a unique attractor $A_\infty \in \mathcal{X}$ such that $\Phi^{(n)}(A) \to A_\infty$ holds for* any *initial set $A \in \mathcal{X}$.*

Proof. Since $(\mathcal{X}, d)$ is a complete metric space and Φ is a contraction, this follows from the Banach fix point theorem. □

Example. The *Koch curve* is generated by the following IFS:

$$\Phi_{\text{Koch}}(A) = \frac{1}{3}A \cup \left[\frac{e^{\frac{i\pi}{3}}}{3}A + \frac{1}{3}\right] \cup \left[\frac{e^{-\frac{i\pi}{3}}}{3}A + \frac{1}{2} + \frac{i}{2\sqrt{3}}\right] \cup \left[\frac{1}{3}A + \frac{2}{3}\right]$$

where $\mathbb{R}^2$ has been identified with $\mathbb{C}$ for notational convenience.

Remark. All results generalize to $\mathbb{R}^n$ for arbitrary dimension n.